

## HYDROMAGNETIC COUETTE FLOW – NUMERICAL METHODS AND COMPARISON WITH EXPERIMENT

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### Abstract

We develop a numerical formulation based on spectral methods suitable to study three dimensional nonlinear hydromagnetic Taylor-Couette flow. The method is used to determine the onset of axisymmetric Taylor vortices, the appearance of wavy modes and the corresponding finite amplitude solutions.

### Introduction

In his landmark 1961 book on stability theory, Chandrasekhar [1] devoted equal attention to the hydrodynamic and the hydromagnetic Couette problems; the latter is the case in which the fluid is a conducting liquid and a magnetic field is applied externally. Despite this early interest in the hydromagnetic Couette problem, which included experiments performed by Donnelly *et al.* [2,3], most of the activity of the following years was devoted to the hydrodynamic case.

The aim of this paper is to investigate effects induced on Couette flow by an externally applied magnetic field. Our work is also motivated by the renewed interest in MHD flows in confined geometries which arises from current and planned experiments to produce dynamo action in the laboratory [4,5]. It must be stressed that our work does not apply directly to the dynamo problem as experiments require rather large Reynolds numbers due to the small magnetic Prandtl number of liquid metals. Nevertheless, it is important to have precise results at small Reynolds number MHD flows in order to develop and test modern acoustic flow visualisation techniques [6], which offer the best chance to detect flow patterns in MHD dynamos. The lack of flow visualisation clearly held back progress in the hydromagnetic Couette problem compared to the hydrodynamic case.

### Formulation

The equations governing incompressible hydromagnetic flow are,

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \frac{1}{\rho \mu_0} (\nabla \wedge \mathbf{B}) \wedge \mathbf{B},$$

$$\partial_t \mathbf{B} = \lambda \nabla^2 \mathbf{B} + \nabla \wedge (\mathbf{u} \wedge \mathbf{B}),$$

plus,  $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{B} = 0$ .

We introduce the dimensionless parameters,

$$\eta = \frac{R_1}{R_2}, \quad \text{Re}_i = \frac{R_i \Omega_i \delta}{\nu}, \quad Q = \frac{\mu_0^2 H^2 \sigma \delta^2}{\rho \nu}, \quad \xi = \frac{\nu}{\lambda},$$

$i=1,2$  for the inner and outer cylinders, and  $\xi$  is the magnetic Prandtl number. A magnetic field, measured by the Hartmann number  $Q$ , is applied externally in the axial direction. The cylinders are assumed to have infinite length.

Donnelly used mercury with Perspex and stainless-steel cylinders. Only a small difference was found between the results. Hereafter we restrict ourselves to the case of insulating boundaries.

To ensure a divergence-free we adopt the toroidal-poloidal decomposition,

$$\mathbf{A} = \psi_0 \hat{\theta} + \varphi_0 \hat{z} + \nabla \wedge (\psi \mathbf{r}) + \nabla \wedge \nabla \wedge (\varphi \mathbf{r}).$$

The potentials,  $\psi, \varphi$ , are then expanded, Fourier in the periodic co-ordinates and by Chebyshev polynomials in the radial direction.

The governing equations for the magnetic field are the  $r$ - components of the induction equation and it's first curl. Although it is commonplace to take the first and second curls for the velocity we proceed by symmetry with the magnetic field and take the  $r$ - component of the momentum equation and it's first curl. In order that we can solve for the pressure we take the divergence of the momentum equation.

No additional boundary conditions are required as we have avoided high order radial derivatives. All five governing equations are only second order in  $r$ . This property makes them easy to timestep accurately (even for fully nonlinear three - dimensional flows) and simplifies implementation enormously. Explicit Adams-Bashforth is used on the nonlinear terms as they are difficult to evaluate and implicit Crank-Nicholson on the linear terms.

## Comparison with experiment

The ratio of the effective viscosity of the flow to the kinematic viscosity of the fluid is equal to  $G/\tilde{G}$  where  $G$  is the torque experienced by the inner cylinder and  $\tilde{G}$  is the component of  $G$  due to the underlying circular-Couette flow (CCF),  $\tilde{u}_\theta = Ar + B/r$ .

Figure 1 shows experimental results by Donnelly and Ozima [2]. Also shown are torques for axisymmetric calculations with their aspect ratio,  $\eta = 0.95$ , and Tabeling's amplitude expansion about the critical point in the narrow gap limit.

As the Reynolds number is increased there is good agreement with Tabeling's amplitude expansion until Donnelly's results deviate from both. Tabeling attributed this to the appearance of wavy modes.

With  $Q = 0$  the onset of wavy modes is not far above the onset of Taylor-vortex flow (TVF). With an imposed axial magnetic field present preliminary results indicate that the onset of wavy modes is indeed significantly inhibited. Field strengths used in

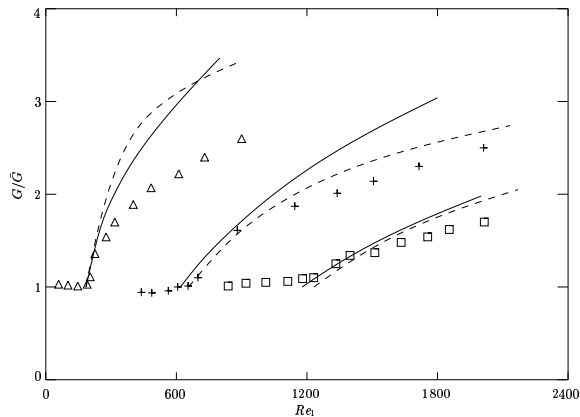


Figure 1: Comparison of torques. Experimental results with  $\eta = 0.95$ .  $\Delta, Q=0$ ;  $+, Q=180$ ;  $\square, Q=652$ . Solid line, numerical results. Dashed line, expansion about the critical point in the narrow gap limit.

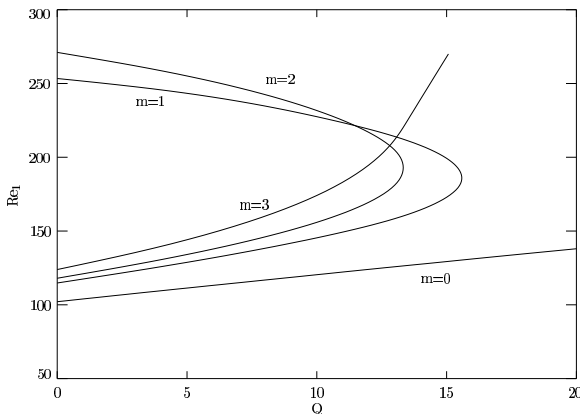


Figure 2: Stability of CCF to axisymmetric perturbations ( $m=0$ ) and of TVF to wavy perturbations ( $m=1,2,3$ ).  $\eta = 0.83$ .

the experiments for investigation of the transition to TVF appear to be rather large given the quick suppression and delicate features we find in the transition to wavy modes (see figure 2).

## Discussion

We have developed a numerical formulation of the governing MHD equations of the cylindrical Couette geometry. It is suitable for timestepping in the nonlinear regime, and agrees well with experiments.

At the cost of coupled equations we avoided the high order derivatives of Marqués' formulation [7], but this appears to help accuracy and stability when using Chebyshev expansions. The method is also second order in time, which seems to be an improvement on Marcus'  $O(\Delta t / Re, \Delta t^2)$  method [8].

Preliminary results indicate that the presence of a magnetic field may have interesting effects, even at relatively low Reynolds numbers, hopefully detectable in both numerical simulations and future experiments.

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