

# On The Influence Of A Bounded Horizontal Temperature On The Onset Of Thermal Convection

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## Abstract

Thermal convection for fluids with a stationary and spatially periodic temperature gradient is performed. Conditions ensuring non-linear global energy stability with respect to the  $L^2$ -norm of the conduction solution are obtained.

## Introduction

Thermal convection in a horizontal fluid layer uniformly heated from below has been studied - in the past as nowadays – by many authors (see, for instance, [1], [5-8], [10], [15-17]). In fact, as it is well known, thermal convection is of fundamental importance in many applications like for example climatology and the diffusion of pollution in the atmosphere.

Recently many researchers have studied this problem on assuming that the layer didn't be uniformly heated from below, but the presence of a spatially constant temperature gradient has been considered [9], [11-14]. Even if this is a more realistic situation, a spatially constant temperature gradient has a (serious) drawback: it implies *at infinity an infinite temperature on the horizontal planes*.

The aim of this paper is to avoid boundedness temperature behaviour on the horizontal planes. To this end we consider the following two realistic situations:

Case i): *the lower plane is maintained to a non-constant temperature greater than the (constant) temperature fixed on the top plane* [2];

Case ii): *both the planes bounding the layer are maintained to a non-constant temperature of which the one of the bottom plane is greater than the one of the top plane* [3].

In both the mentioned situations, we consider a stationary spatially periodic temperature gradient on the bounding planes of the layer, which implies a bounded temperature.

The plan of the paper is the following. Section 2 is dedicated to introduce thermal convection problem of a fluid in an infinite horizontal layer (Bènard problem) under the assumption that at once the lower plane bounding the layer is not uniformly heated, but is maintained to a spatially periodic temperature – in the horizontal direction – such that the temperature

difference between the lower and the upper plane is positive. Then in Section 3 we perform non-linear stability analysis through the Liapunov direct method and global (non-linear) stability results are obtained. Finally we show two numerical tables containing the energy stability thresholds versus horizontal temperature gradient.

## Statement of the problem

Let be  $\{(x, y, z) \in \mathfrak{R}^3: 0 \leq z \leq d\}$  a horizontal layer filled by a newtonian and incompressible fluid. The dimensionless equations governing the motion of the fluid in the horizontal layer  $\{(x, y, z) \in \mathfrak{R}^3: 0 \leq z \leq 1\}$  are

$$\begin{cases} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \Delta \mathbf{v} + R(T - T_0) \mathbf{k} \\ \nabla \cdot \mathbf{v} = 0 \\ \text{Pr} (T_t + \mathbf{v} \cdot \nabla T) = \Delta T \end{cases} \quad (1)$$

where  $z$  is the upward vertical,  $\mathbf{v}$  is the velocity field,  $T$  is the temperature,  $T_0$  is a reference temperature,  $p$  is the reduced pressure and finally  $R$  and  $\text{Pr}$  are dimensionless parameter defined as follows:

$$R^2 = \frac{\alpha g \Delta T d^3}{\kappa n} \quad (\text{Rayleigh number})$$

$$\text{Pr} = \frac{\kappa}{n} \quad (\text{Prandtl number})$$

where  $\alpha$  is the thermal expansion coefficient,  $g$  is the gravity,  $\Delta T (>0)$  is the constant temperature difference between the points belonging to the bottom plane and that ones belonging to the top plane at  $x=0$ ,  $v$  is the kinematics viscosity,  $d$  is the depth of the layer and  $\kappa$  is the thermal diffusivity. To the system (1) we append the boundary conditions

$$T(z=0)=[\alpha^*(e^2-1)/(2e)]\sin x+T_1+1, \quad T(z=1)=T_1 \quad (2)$$

with  $\alpha^* \in (0, 0.64)$  in the case i) mentioned in the introduction, or the following boundary conditions

$$T(z=0)=\alpha^* \sin x+T_1+1, \quad T(z=1)=\alpha^*/e \sin x+T_1 \quad (3)$$

with  $\alpha^* \in (0, 1)$  if we consider the case ii) mentioned in the introduction.

The boundary value problem (1), (2) admits the following motionless state (conduction solution) [4]

$$m_1=\{\mathbf{v}=\mathbf{0}; T_b(x,z)=\alpha^* \sinh(1-z)\sin x-z+T_1+1; p_b\} \quad (4)$$

while the boundary value problem (1), (3) admits the following motionless state (conduction solution)

$$m_2=\{\mathbf{v}=\mathbf{0}; T_b(x,z)=\alpha^* e^{-z}\sin x-z+T_1+1; p_b\}. \quad (5)$$

Let us underline that - in both cases - the basic temperature field  $T_b(x,z)$  exhibits a periodic behaviour in the  $x$  direction and hence we are considering a bounded temperature in horizontal direction. Since we want to study the stability of  $m_1$  and  $m_2$ , let us consider  $\mathbf{u}, \theta, \pi$  the dimensionless perturbation to the velocity, temperature and pressure fields, respectively. The dimensionless equations governing the evolution of the perturbation fields in the strip  $\mathcal{R}^2 \times [0,1]$  are

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \Delta \mathbf{u} + R \mathbf{J} \mathbf{k} \\ \nabla \cdot \mathbf{u} = 0 \\ \text{Pr}(\mathbf{J}_t + \mathbf{u} \cdot \nabla \mathbf{J}) = \Delta \mathbf{J} - R \mathbf{u} \cdot \nabla T_b(x,z) \end{cases} \quad (6)$$

To the previous system we add the following free-free boundary conditions:

$$w = \frac{\partial^2 w}{\partial z^2} = \Delta w = 0, \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0, \quad \mathbf{J} = 0 \quad (7)$$

where  $\mathbf{u} = (u, v, w)$ . In the sequel, as usual, we shall assume that the perturbation fields are periodic functions of  $x$  and  $y$  of periods  $2\mathbf{p}/a_x, 2\mathbf{p}/a_y$ . We

shall denote by  $\Omega$  the periodicity cell and by  $\langle \cdot \rangle, \|\cdot\|$  the integral and the  $L^2$ -norm on  $\Omega$ . Finally, since the stability of  $m_i$  ( $i=1,2$ ) makes sense only in a class of solutions of (6)-(7) in which  $m_i$  ( $i=1,2$ ) is unique, we exclude any other rigid motion on requiring that  $\langle u \rangle = \langle v \rangle = 0$ .

## Non-linear stability

In this section we perform the nonlinear energy stability analysis of the conduction solution in both cases i) and ii). To this end let us introduce the following Liapunov functional

$$V(t) = \frac{1}{2} \|\mathbf{u}\|^2 + \frac{\text{Pr}}{2} \|\mathbf{q}\|^2. \quad (8)$$

On evaluating the time derivative of  $V(t)$  along the solutions of (6)-(7), by virtue of some well-known inequalities, the following theorems hold true.

**Theorem 1.** *Let be  $R_B=657.511$  the stability threshold of the classical Bènard problem with free-free boundary conditions [5, 6], [8], [16]. On setting*

$$R_E = \frac{2\mathbf{p}^2 R_B}{2\mathbf{p}^2 + \mathbf{a}^* (A+B)R_B} \quad (9)$$

where  $A=1.1752$  and  $B=1.54308$ , the condition

$$R < R_E \quad (10)$$

ensures the asymptotic, exponential, global non-linear stability of the conduction solution  $m_1$  with respect to the  $V$ -norm, according to the following inequality

$$V(t) \leq V(0) \exp\left\{ \mathbf{g} \frac{R - R_E}{R_E} t \right\}, \quad t \geq 0, \quad \mathbf{g} = \text{const} > 0.$$

**Theorem 2.** *Let be  $R_B=657.511$  the stability threshold of the classical Bènard problem with free-free boundary conditions [5, 6], [8], [16]. On setting*

$$R_E^* = \frac{\mathbf{p}^2 R_B}{\mathbf{p}^2 + \mathbf{a}^* R_B} \quad (11)$$

the condition

$$R < R_E^* \quad (12)$$

ensures the asymptotic, exponential, global non-linear stability of the conduction solution  $m_2$  with respect to the  $V$ -norm, according to the following inequality

$$V(t) \leq V(0) \exp\left\{ \mathbf{g} \frac{R - R_E^*}{R_E^*} t \right\}, \quad t \geq 0, \quad \mathbf{g} = \text{const} > 0.$$

In the following tables we compare the energy stability thresholds obtained, with that ones linked to the maximum (positive) temperature difference between the lower and the upper planes bounding the layer.

Case i). Let be

$$\bar{R}^{-2} = R^2 \left( 1 + \mathbf{a}^* \frac{e^2 - 1}{2e} \right)$$

Table 1: Critical Rayleigh numbers in the case i)

$\alpha^*$	$R_E^2$	$\overline{R_E^2}$
0	657.511	657.511
0.1	359.113	401.316
0.3	155.039	209.700
0.5	85.967	136.481

Case ii). Let be

$$\overline{R}^{*2} = R^{*2} \left( 1 + a^* \frac{e-1}{e} \right)$$

Table 2: Critical Rayleigh numbers in the case ii)

$\alpha^*$	$R_E^{*2}$	$\overline{R_E^{*2}}$
0	657.511	657.511
0.1	414.280	440.460
0.3	207.656	247.310
0.5	124.397	163.859

**Remark 1** – *Let us underline that, as it immediately follows on reading the Tables 1 and 2, the case i) is less stable than the case ii). In fact, accounting for the boundary conditions (2) and (3), it arises that the (absolute) temperature difference between the lower and the upper plane in the case i) is greater than in the case ii).*

Finally, we will study in particular the case of bi-dimensional perturbations to the velocity fields, on obtaining better non-linear stability thresholds of the conduction solutions [3].

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