

STABILITY OF A TWO-LAYER DEAN FLOW WITH A CAPILLARY LIQUID-LIQUID INTERFACE

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Abstract

Three-dimensional stability of a two-layer Dean flow in a cylindrical annulus is studied numerically using a global Galerkin method. A simplified model corresponding to large inner radius of the annulus is considered. It is shown that the instability becomes three-dimensional if one of the fluid layers is thin and that the instability onset is not affected by possible small deformations of the interface. Stability diagrams and patterns of the three-dimensional perturbations are reported. It is shown that the primary instability switches from steady axisymmetric to oscillatory three-dimensional when one of the layers is relatively thin. It is concluded that even when the axisymmetric perturbation is the most dangerous, the resulting supercritical flow is expected to be three-dimensional. Possible multiplicity of supercritical three-dimensional states is predicted. The results are of potential importance for development of novel bioseparators employing Dean vortices for enhancement of mass transfer of a passive scalar (say, a protein) through the interface.

Introduction

The Dean vortices, arising in curved channels [1], are known as a means for intensification of heat [2] and mass transfer [3] in single-phase liquids. Recently, certain attempts were made to intensify mass transfer of a passive scalar through a boundary separating two immiscible liquids by creating spatially periodic vortical flows inside both liquid phases using a two-fluid Taylor-Couette apparatus [4]. Obviously, the Dean flow is another possible originator of the vortical flow inside a two-fluid system.

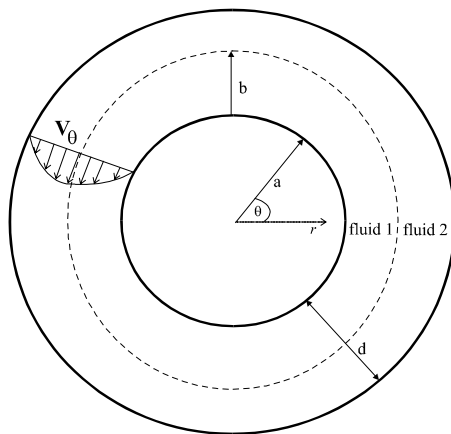


Fig.1. Sketch of the problem

In the present work we study the onset of Dean vortices in the two-fluid Dean problem (Fig.1). Namely, stability of a flow driven by a constant azimuthal pressure gradient G in a cylindrical annulus filled with

two immiscible liquid layers is considered. The liquid-liquid interface can be either non-deformable or deformable and subject to capillary forces. In the latter case small perturbations of the interface are included in the formulation of the stability problem. The gravity effect is disregarded.

The stability problem is solved using an extension of the global Galerkin approach [5]. This approach uses linear combinations of Chebyshev polynomials to satisfy analytically no-slip or stress-free boundary conditions. In the present work the Galerkin basis is constructed for a two-fluid case such that not only no-slip conditions, but also those of continuity and the balance of viscous stresses at the liquid-liquid interface are satisfied a priori. The balance of the capillary and normal viscous stresses is included in the numerical model assuming small deformations of the interface [6]. The proposed approach can be used for stability analysis of various two-fluid flows [7], e.g. for two-layer Rayleigh-Bénard [8] or Taylor-Couette [9] flows.

Two main conclusions are derived from the stability analysis, which accounts for three-dimensional perturbations and for possible small deformations of the liquid-liquid interface. First, the instability observed in the system is not affected by deformations of the interface and corresponds to onset of vortical motion in both layers. Second, the instability switches from axisymmetric to three-dimensional when one of the layers becomes thin (i.e., b approaches zero or to the gap thickness d). Switches between different perturbation modes and spatial patterns of the non-axisymmetric perturbations are illustrated, using a representative example. On the basis of the calculated

stability diagrams, we predict existence of multiple three-dimensional supercritical flow states, which appear as spiraling waves that rotate about the axis and propagate along it.

Formulation of the problem

Consider a cylindrical annulus, whose radius varies in the interval $a \leq r \leq a+d$, filled with two immiscible Newtonian incompressible liquids 1 and 2 which, in the unperturbed state, occupy cylindrical layers $a \leq r \leq a+b$ and $a+b \leq r \leq a+d$, respectively (see Fig.1). It is assumed that the flow is driven by a constant azimuthal pressure gradient $\partial p / \partial \theta = G = \text{const}$. The flow is described by the dimensionless Navier-Stokes and Continuity equations

$$\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \nabla) \mathbf{v}_i = -\frac{1}{\rho_{1i}} \frac{1}{r} \mathbf{e}_\theta - \frac{1}{\rho_{1i}} \nabla P_i + \frac{\eta_{1i}}{\rho_{1i}} \frac{1}{Re} \Delta \mathbf{v}_i \quad (1)$$

$$\nabla \cdot \mathbf{v}_i = 0 \quad (2)$$

where $\mathbf{v}_i = \{U_i, V_i, W_i\}$ is the flow velocity, P_i the pressure additional to the component responsible for the azimuthal pressure gradient of liquid i . U_i , V_i and W_i denoting the radial, azimuthal and axial velocities, respectively, and the subscript i referring to liquids 1 and 2. The governing parameters are the Reynolds number $Re = \rho_1 d v_0 / \eta_1 = d \sqrt{G \rho_1} / \eta_1$, and density and viscosity ratios $\rho_{1i} = \rho_i / \rho_1$, $\eta_{1i} = \eta_i / \eta_1$. For convenience of comparison with the previous studies [1-3,10] we introduce the Dean number $De = Re / \sqrt{\bar{a}}$ which accounts also for the radius of curvature of the cylindrical annulus ($\bar{a} = a/d$). No-slip boundary conditions are imposed on the boundary of the annulus.

To simplify the problem we replace the radial coordinate r by $r = \bar{a} + x$. Then, assuming $\bar{a} \gg 1$ we replace $r = \bar{a} + x$ by \bar{a} . After this simplification the basic dimensionless flow field is given by

$$V_1 = \frac{Re}{2\bar{a}} x^2 + \frac{\eta_{12}}{\rho_{12}} \frac{Re}{2\bar{a}} \frac{\bar{b}^2 (\rho_{12}/\eta_{12} - 1) - \rho_{12}/\eta_{12}}{\bar{b}(\eta_{12}/\rho_{12} - 1) + 1} x \quad (3)$$

$$V_2 = \frac{\rho_{12}}{\eta_{12}} \frac{Re}{2\bar{a}} x^2 + \frac{Re}{2\bar{a}} \frac{\bar{b}^2 (\rho_{12}/\eta_{12} - 1) - \rho_{12}/\eta_{12}}{\bar{b}(\eta_{12}/\rho_{12} - 1) + 1} x - \frac{Re}{2\bar{a}} \frac{\bar{b}(1 - \rho_{12}/\eta_{12}) + \bar{b}^2 (\rho_{12}/\eta_{12} - 1)}{\bar{b}(\eta_{12}/\rho_{12} - 1) + 1} \quad (4)$$

where $\bar{b} = b/d$ and $x = (r - a)/d$ and. The flow (3),(4) is an extension of the basic flow of Dean [10] for a two-liquid system.

The present stability analysis accounts for three-dimensional perturbations of the basic flow (1),(2) and for possible small deformations of the liquid-liquid interface $r = a + b$. The perturbations of the velocity,

pressure and the liquid-liquid interface are represented as $\mathbf{u}_i = (u_i(x), v_i(x), w_i(x)) \exp[in\theta + ikz + \lambda t]$, $p_i = p_i(x) \exp[in\theta + ikz + \lambda t]$, and $\xi = \delta \exp[in\theta + ikz + \lambda t]$, respectively. Here the real number k is the axial wavenumber, and the integer number n the azimuthal wavenumber. After elimination of w_i and p_i the linear stability problem reduces to the following equations:

$$\lambda D u_i = R_i D^2 u_i - \frac{in}{\bar{a}} V_i D u_i - \frac{in}{\bar{a}} \frac{dV_i}{dx} \frac{du_i}{dx} - \frac{2k^2}{\bar{a}} V_i v_i \quad (5)$$

$$\lambda \left[v_i - \frac{in}{\bar{a}k^2} \frac{du_i}{dx} \right] = R_i D v_i - \frac{dV_i}{dx} u_i - \frac{in}{\bar{a}k^2} \frac{\eta_{1i}}{\rho_{1i}} \frac{1}{Re} \frac{d}{dx} D u_i - \frac{n^2 V_i}{\bar{a}^2 k^2} \frac{du_i}{dx} - \frac{in V_i}{\bar{a}} v_i - \frac{V_i}{\bar{a}} u_i \quad (6)$$

where $D = d^2/dx^2 - k^2$. For the deformable interface Eqs.(5),(6) have to be solved with the following boundary conditions:

$$\text{at } x = 0: \quad u_1 = v_1 = du_1/dx = 0 \quad (7)$$

$$\text{at } x = 1: \quad u_2 = v_2 = du_2/dx = 0 \quad (8)$$

$$\text{at } x = \bar{b}: \quad u_1 = u_2 \quad (9)$$

$$v_1 = v_2 \quad (10)$$

$$du_1/dx = du_2/dx \quad (11)$$

$$d^2 u_1/dx^2 + k^2 u_1 = \eta_{12} \left[d^2 u_2/dx^2 + k^2 u_2 \right] \quad (12)$$

$$\lambda \left[\rho_{12} \frac{du_2}{dx} - \frac{du_1}{dx} \right] = \frac{\bar{\delta} k^2}{\bar{a}} (\rho_{12} - 1) V_1^2 + \frac{1}{Re} \left[\frac{dD}{dx} - 2 \frac{d}{dx} \right] (\eta_{12} u_2 - u_1) - \quad (13)$$

$$- \rho_{12} \frac{in}{\bar{a}} V_1 \frac{d}{dx} (\rho_{12} u_2 - u_1) - \frac{\bar{\delta} k^2}{We} \left[\frac{1-n^2}{\bar{b}^2} - k^2 \right]$$

$$\lambda \bar{\delta} = u_1 - \frac{in}{\bar{b}} V_1 \bar{\delta} \quad (14)$$

In the case of non-deformable interface the boundary conditions (13) and (14) should be omitted, and the boundary condition (9) at $x = \bar{b}$ should be replaced by $u_1 = u_2 = 0$.

Results

The stability analysis was carried out for the fixed values $\rho_{12} = \eta_{12} = 1.1$ and \bar{b} varying from 1 to 9. Two main conclusions were drawn. First, the instability observed in the system corresponds to onset of vortical motion and is not affected by deformations of the interface. Second, the instability is axisymmetric when

the depth of the inner layer b lies in the interval $0.24 < b/d < 0.87$. When one of the layers is thin (i.e., b is close to zero or to the gap thickness d , or \bar{b} is close to zero or to unity), the instability is caused by a non-axisymmetric perturbation characterized by a relatively high azimuthal wave number. The latter is illustrated in Figs.2-4 where the dependence of marginal Dean number $De = Re/\sqrt{\bar{a}}$ on the axial wavenumber k is shown for different azimuthal numbers n .

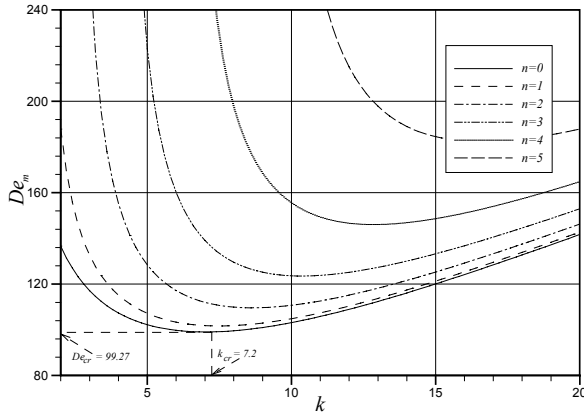


Fig.2. Stability diagram for $\rho_{12} = \eta_{12} = 1.1$, $\bar{a} = 10$ and $\bar{b} = 0.5$.

At $\bar{b} = 0.5$ the lowest De_m corresponds to the axisymmetric azimuthal mode with $n_{cr} = 0$ (Fig.2), while at $\bar{b} = 0.2$ and 0.9 the lowest De_m correspond to the non-symmetric modes with $n_{cr} = 12$ and 7 , respectively. Note that axisymmetric instability is steady (like in the classical case considered by Dean [10]), while the three-dimensional instability is oscillatory.

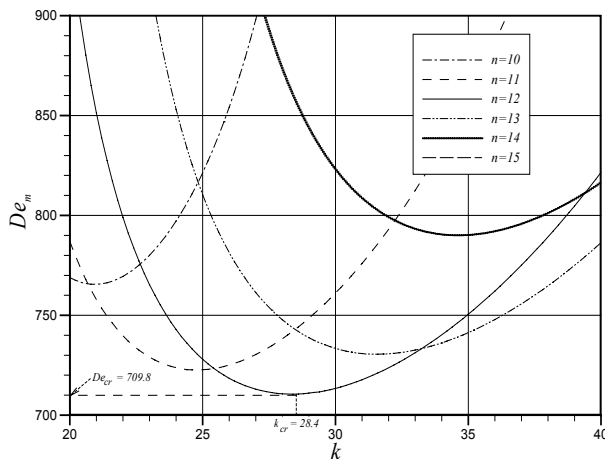


Fig.3. Stability diagram for $\rho_{12} = \eta_{12} = 1.1$, $\bar{a} = 10$ and $\bar{b} = 0.2$. Deformable interface. Marginal Dean

numbers corresponding to the modes $n < 10$ and $n > 15$ are above the value $De = 900$.

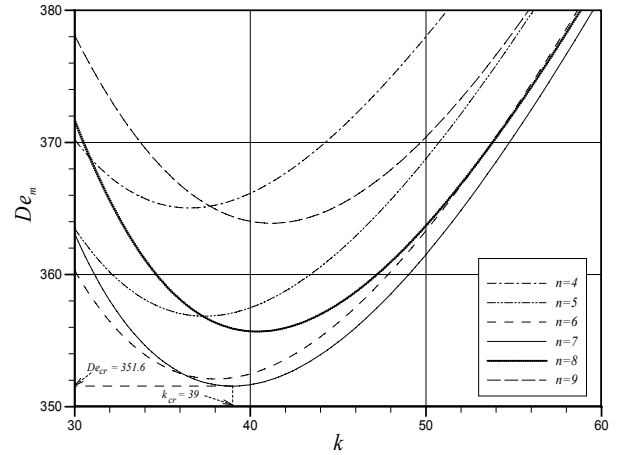


Fig.4. Stability diagram for $\rho_{12} = \eta_{12} = 1.1$, $\bar{a} = 10$ and $\bar{b} = 0.9$. Deformable interface. Marginal Dean numbers corresponding to the modes $n < 4$ and $n > 9$ are above the value $De = 380$.

The stability diagrams obtained (like those shown in Figs. 2 and 3) show that there exist several azimuthal modes whose marginal Dean numbers are rather close. This means that at the supercritical Dean numbers one can expect multiple three-dimensional flow states, which appear as spiraling waves that rotate about the axis and propagate along it. Patterns of three-dimensional perturbations are illustrated in Fig.5, where radial coordinate is zoomed by the factor of 20.

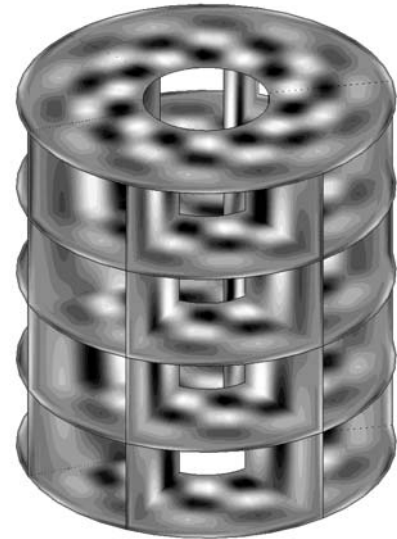


Fig.5a. Perturbation of the radial velocity. $\bar{b} = 0.9$, $De_{cr} = 351.5$, $n_{cr} = 7$, $k_{cr} = 39$, $\omega_{cr} = 0.576$.

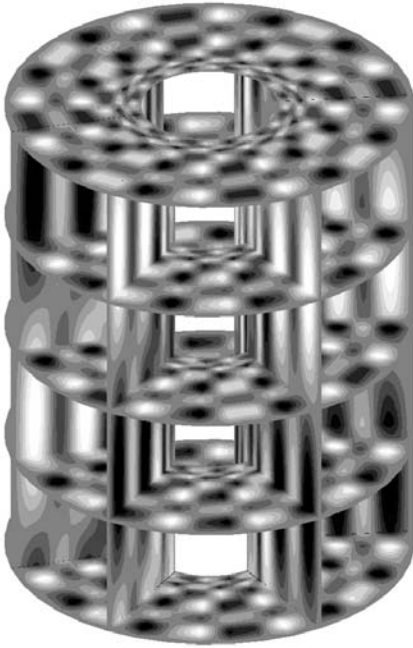


Fig.5b. Perturbation of the radial velocity. $\bar{b} = 0.2$, $De_{cr} = 710.5$, $n_{cr} = 12$, $k_{cr} = 28.2$, $\omega_{cr} = 1.31$.

Conclusions

The stability of two-fluid Dean flow was studied numerically using the global Galerkin method. The numerical approach of [5] was extended to implement all the boundary conditions (including those imposed on the liquid-liquid interface) into Galerkin basis functions. A special numerical treatment of small deformations of the liquid-liquid interface, including capillary effects was developed.

The onset of the instability, which manifests itself in appearance of axisymmetric or three-dimensional vortices, was studied for different widths of the fluid layers. For a fixed inner radius of the annulus ($\bar{a} = 10$) and fixed viscosity and density ratios ($\rho_{12} = \eta_{12} = 1.1$) it was shown that the instability is axisymmetric when the relative depth of the inner layer lies in the range $0.24 < \bar{b} < 0.97$, and three-dimensional otherwise. At the same time, onset of the instability (axisymmetric as well as three-dimensional) is also characterized by development of several three-dimensional modes, which become unstable already at very small supercriticalities. Therefore, the resulting supercritical state should be expected to be three-dimensional. Moreover, multiplicity of stable supercritical states seems to be possible.

Acknowledgements

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