

MHD INSTABILITY IN CYLINDRIC TAYLOR-COUPETTE FLOW

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Abstract

The linear marginal instability of an axisymmetric MHD Taylor-Couette flow of infinite vertical extension is considered. For flows with a resting outer cylinder there is a well-known characteristic Reynolds number even without magnetic field but for sufficiently weak magnetic fields there are solutions with *smaller* Reynolds numbers so that a characteristic minimum exists. The minimum only exists, however, for not too small magnetic Prandtl numbers. For small magnetic Prandtl numbers one only finds the typical magnetic suppression of the instability.

We are here particularly interested in the case where the outer cylinder rotates so fast that the Rayleigh criterion for *hydrodynamic* stability is fulfilled. We find that for given magnetic Prandtl number now always a magnetic field amplitude exists where the characteristic Reynolds number is minimal. These critical values are computed for different magnetic Prandtl numbers. In all cases the Reynolds numbers are running with $1/\text{Pm}$ so that for the small magnetic Prandtl numbers of sodium (10^{-5}) or gallium (10^{-6}) the critical Reynolds numbers exceed values of 10^6 or 10^7 , resp.

Introduction

The longstanding problem of the generation of turbulence in various hydrodynamically stable situations has found a solution in recent years with the so called ‘Balbus-Hawley instability’, in which the presence of a magnetic field has a destabilizing effect on a differentially rotating flow, provided that the angular velocity decreases outwards with the radius. This magnetorotational instability (MRI) has been discovered decades ago for ideal Couette flow, but it has never been observed in the laboratory. Moreover, Chandrasekhar (1961) already suggested the existence of MRI for ideal Taylor-Couette flow, but his results for non-ideal fluids for small gaps and within the small magnetic Prandtl number approximation demonstrated the absence of MRI for hydrodynamically stable and/or unstable flow. Recently, Goodman and Ji (2001) claimed that this absence of MRI was due to the use of the small magnetic Prandtl number limit. The magnetic Prandtl number $\text{Pm} = \nu/\eta$ is really very small under laboratory conditions. Obviously, the understanding of this phenomenon is very important for possible experiments, Taylor-Couette flow dynamo experiments included.

Here the dependence of the magnetic Prandtl number on the MHD instability of Taylor-Couette flow is investigated. The simple model of uniform

density fluid contained between two vertically-infinite rotating cylinders is used with constant magnetic field parallel to the rotation axis. For viscous flows then the most general form of Ω is

$$\Omega(r) = a + b/R^2, \quad (1)$$

where a and b are two constants related to the angular velocities Ω_{in} and Ω_{out} with which the inner and the outer cylinders are rotating. If R_{in} and R_{out} ($R_{\text{out}} > R_{\text{in}}$) are the radii of the two cylinders then

$$a = \Omega_{\text{in}} \hat{\eta}^2 \frac{\hat{\mu} - \hat{\eta}^2}{1 - \hat{\eta}^2} \quad \text{and} \quad b = \Omega_{\text{in}} R_{\text{in}}^2 \frac{(1 - \hat{\mu})}{1 - \hat{\eta}^2}, \quad (2)$$

with

$$\hat{\mu} = \Omega_{\text{out}}/\Omega_{\text{in}} \quad \text{and} \quad \hat{\eta} = R_{\text{in}}/R_{\text{out}}. \quad (3)$$

After the Rayleigh stability criterion, $d(R^2\Omega)/dR > 0$, rotation laws with positive numbers a are hydrodynamically stable, i.e. for $\hat{\mu} > \hat{\eta}^2$. Taylor-Couette flows with resting outer cylinders ($\hat{\mu} = 0$) are thus never stable.

Boundary conditions

An appropriate set of ten boundary conditions is needed to solve the equation system (see Rüdiger and Zhang 2001). Always no-slip conditions for the velocity on the walls are used, i.e.

$$u_R = 0, \quad u_\phi = 0, \quad \frac{du_R}{dR} = 0. \quad (4)$$

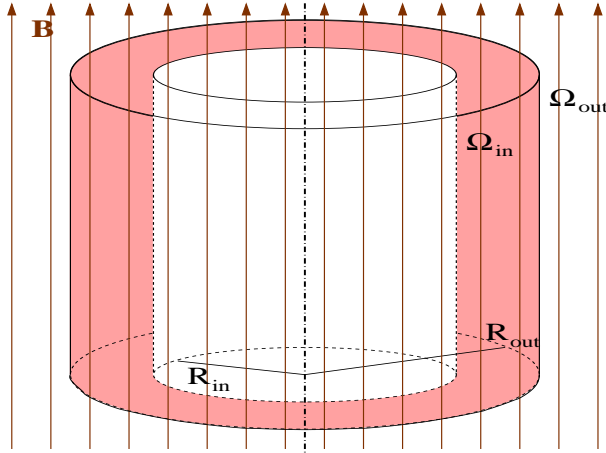


Figure 1: Cylinder geometry of the Taylor-Couette flow

The magnetic boundary conditions depend on the electrical properties of the walls. The transverse currents and perpendicular component of magnetic field should vanish on conducting walls, hence

$$\frac{db_\phi}{dR} + \frac{b_\phi}{R} = 0, \quad b_R = 0. \quad (5)$$

The above boundary conditions (4) and (5) are valid for $R = R_{in}$ and for $R = R_{out}$.

Rotation and magnetic field are represented in the numerical simulations by the Reynolds number $Re = \Omega_{in} R_{in} (R_{out} - R_{in}) / \nu$ and the Hartmann number $Ha = (R_{in} (R_{out} - R_{in}) / \mu_0 \rho \nu \eta)^{1/2} B$.

Resting outer cylinder

In Fig. 2 a resting outer cylinder is considered for a medium-size gap of $\hat{\eta}=0.5$ and for $Pm=1$. As we know for vanishing magnetic field the exact Reynolds number for this case is about 68 – well represented by the result for $Ha=0$ in Fig. 2. But for increasing magnetic field the Reynolds number is reduced so that the excitation of the Taylor vortices becomes easier than without magnetic field. The minimum Reynolds number Re_{crit} of about 52.5 for $Pm=1$ is reached for $Ha_{crit} \simeq 6...7$. This magnetic induced subcritical excitation of Taylor vortices is due to the MRI. Always for a (say) critical Hartmann number the Reynolds numbers take a minimum which we shall call the critical Reynolds numbers. For even stronger magnetic fields – as it must be – the magnetic field starts to suppress the instability (see also Rüdiger and Zhang 2001). In Fig. 3 the same container is considered but for the small magnetic Prandtl number of 10^{-5} . The minimum characteristics for $Pm=1$

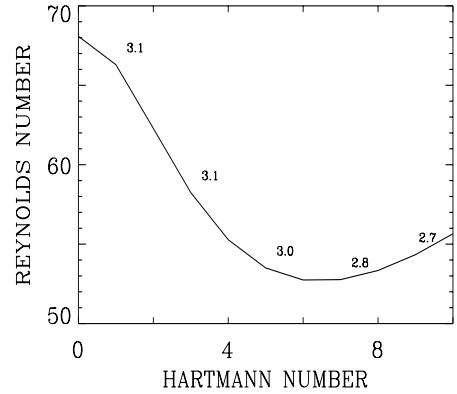


Figure 2: The Reynolds numbers for Taylor-Couette flow with resting outer cylinder with $\hat{\eta} = 0.5$ and for $Pm=1$. There is instability even without magnetic fields but its excitation is much easier with magnetic fields with Hartmann numbers of about 6...7. The line is marked with those wavenumbers for which the eigenvalues are minimal.

completely disappears, only suppression of the instability by the magnetic field can be observed.

Rotating outer cylinder

Another situation holds if the outer cylinder rotates so fast that the rotation law does no longer fulfill the Rayleigh criterion and a solution for $Ha=0$ cannot exist. Then the nonmagnetic eigenvalue along the vertical axis moves to infinity and we should always have a minimum. It is the basic situation in astrophysical applications such for accretion disks with a Kepler rotation law. Here

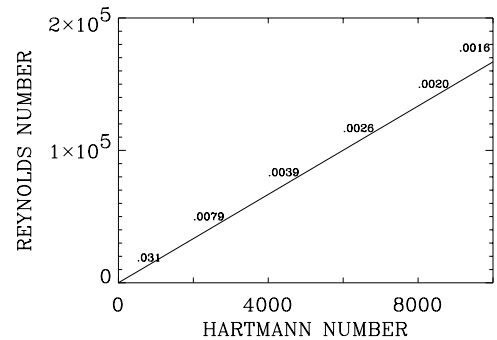


Figure 3: The same as in Fig. 2 but for $Pm=10^{-5}$. The minimum characteristic for $Pm=1$ completely disappears.

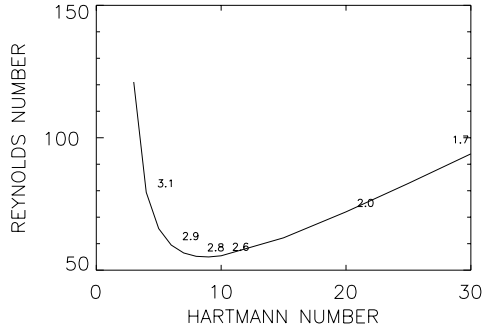


Figure 4: MHD instability map for Taylor-Couette flow for $\hat{\eta} = 0.5$ and $\text{Pm}=1$. The outer cylinder rotates with 33% of the rotation rate of the inner cylinder so that after the Rayleigh criterion the hydrodynamic instability disappears. Again the curve is marked with the critical wavenumbers

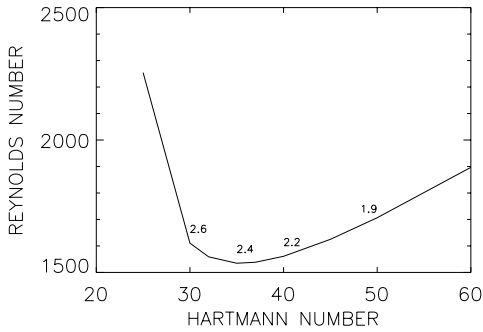


Figure 5: The same as in Fig. 4 but for $\text{Pm}=0.01$

the question is whether the critical Reynolds number and the critical Hartmann number can experimentally be realized. The Figs. 4..6 present the results for both various Hartmann numbers and magnetic Prandtl numbers for a medium-sized gap of $\hat{\eta}=0.5$.

There are always minima of the characteristic Reynolds numbers for certain Hartmann numbers. The minima and the critical Hartmann numbers increase for decreasing magnetic Prandtl numbers. For $\hat{\eta}=0.5$ and $\hat{\mu}=0.33$ the critical Reynolds numbers together with the critical Hartmann numbers are plotted in Fig. 7.

For the small magnetic Prandtl numbers we find interesting and simple relations. With

$$C_{\Omega} = \text{RePm} \quad (6)$$

and

$$\text{Ha}^* = \text{Ha}\sqrt{\text{Pm}} \quad (7)$$

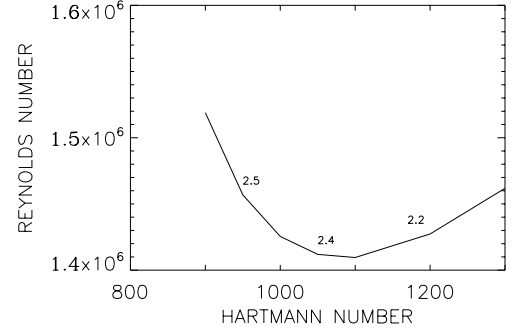


Figure 6: The same as in Fig. 4 but for $\text{Pm}=10^{-5}$

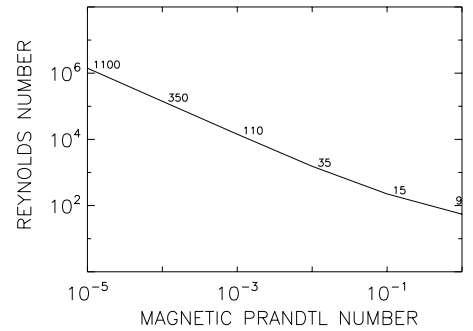


Figure 7: The critical Reynolds numbers for given Prandtl numbers marked with those Hartmann numbers where the Reynolds number is minimal.

it follows

$$C_{\Omega} \simeq 14 \quad (8)$$

and

$$\text{Ha}^* \simeq 3.3. \quad (9)$$

C_{Ω} is the magnetic Reynolds number, $C_{\Omega} = \Omega_{\text{in}} R_{\text{in}} (R_{\text{out}} - R_{\text{in}}) / \eta$ (or dynamo number) and Ha^* is the magnetic Hartmann number $\text{Ha}^* = (R_{\text{in}} (R_{\text{out}} - R_{\text{in}}) / \mu_0 \rho \eta^2)^{1/2} B$.

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