

RAYLEIGH-BÉNARD INSTABILITY IN ANISOTROPIC MAGNETOHYDRODYNAMICS

Presenting Maiellaro, M.¹, Labianca, A.¹

¹Univ. & Politechn. of Bari, Math. Dept., v. Orabona, 4 I-70125 Bari - Italy

Abstract

Conditions ensuring the validity of the principle of exchange of stabilities in the anisotropic magnetohydrodynamic Rayleigh-Bénard problem are obtained. The behaviour of the Rayleigh critical curves is studied and the instability- stability analysis is performed.

Introduction

The equations governing the evolution of a thermal-electro-conducting fluid in the anisotropic MHD [1] [2] and in the Boussinesq approximation [3] are:

$$\left\{ \begin{array}{l} \mathbf{v}_t = -\mathbf{v} \cdot \nabla \mathbf{v} - \frac{1}{\rho} \nabla \pi + \nu \Delta_2 \mathbf{v} + \\ \quad + \frac{\mu_e}{\rho} \mathbf{H} \cdot \nabla \mathbf{H} + \mathbf{g} [1 - \alpha(T - T_0)] \\ \mathbf{H}_t = \nabla \times (\mathbf{v} \times \mathbf{H}) + \eta \Delta_2 \mathbf{H} + \\ \quad + \beta_1 \nabla \times (\mathbf{H} \times \nabla \times \mathbf{H}) + \\ \quad + \beta_2 \nabla [\mathbf{H} \times (\mathbf{H} \times \nabla \times \mathbf{H})] \\ T_t = -\mathbf{v} \cdot \nabla T + k \Delta_2 T \\ \nabla \cdot \mathbf{v} = 0 \\ \nabla \cdot \mathbf{H} = 0 \end{array} \right. \quad (1)$$

where

- $\pi = p + \frac{1}{2} \mu_e H^2 =$ total pressure
- $\mathbf{v} =$ kinetic field
- $\mathbf{H} =$ magnetic field
- $T =$ thermal field
- $\mathbf{g} =$ gravity field
- $\rho =$ density
- $\nu =$ viscosity
- $\mu_e =$ magnetic permeability
- $\alpha =$ thermal expansion coefficient
- $\eta =$ magnetic viscosity
- $T_0 =$ reference temperature
- $k =$ thermal conductivity
- $\beta_1 =$ Hall coefficient
- $\beta_2 =$ ion-slip coefficient
- $F_t =$ local time derivative

Denoting by $\{O, \mathbf{e}_i\} = Oxyz$, ($i = 1, 2, 3$) an orthonormal frame of reference with the z -axis vertical positive upwards, we assume that:

- i) the fluid fills the horizontal layer $0 \leq z \leq d$, and is embedded in an external magnetic field $\mathbf{H}_0 = H_0 \mathbf{e}_3$,
- ii) the boundaries $z = 0$ and $z = d$ are electrically insulated and at fixed temperatures given by T_0 and T_d ($< T_0$) respectively.

As it is well known [3] the Bénard problem concerns the stability-instability of the conduction solution

$$\left\{ \begin{array}{l} \tilde{\mathbf{v}} = \mathbf{0}, \tilde{\mathbf{H}} = \mathbf{H}_0, \tilde{T} = T_0 - (T_0 - T_d)z/d, \\ \tilde{p} = p_2(z) \end{array} \right. \quad (2)$$

where $p_2(z)$ is obtained integrating (1) with $\left\{ \mathbf{v} = \tilde{\mathbf{v}}, \mathbf{H} = \tilde{\mathbf{H}}, T = \tilde{T} \right\}$.

Denoting by $\mathbf{u}, \mathbf{h}, \theta, p$ the perturbations to the kinetic, magnetic, thermal and pressure fields respectively, by introducing the star shaped nondimensional variables and fields

$$\begin{aligned} x &= x^* d; t = t^* d^2 / \eta; \mathbf{u} = \mathbf{u}^* \eta / d; \\ \mathbf{h} &= \mathbf{h}^* H_0; \theta = \theta^* (T_0 - T_d) \eta / k \end{aligned}$$

dropping the stars and linearizing we obtain

$$\left\{ \begin{array}{l} \mathbf{u}_t = -\nabla \mathcal{P} + \sigma^2 M^2 \mathbf{e}_3 \cdot \nabla \mathbf{h} + \sigma^2 \Delta_2 \mathbf{u} + \\ \quad + R_a^m \theta \mathbf{e}_3 \\ \mathbf{h}_t = \mathbf{e}_3 \cdot \nabla \mathbf{u} + \Delta_2 \mathbf{h} + \beta_H \nabla \times (\mathbf{e}_3 \times \nabla \times \mathbf{h}) + \\ \quad + \beta_I \nabla \times [\mathbf{e}_3 \times (\mathbf{e}_3 \times \nabla \times \mathbf{h})] \\ \mathcal{P}_r^m \theta_t = u_3 + \Delta_2 \theta \\ \nabla \cdot \mathbf{u} = 0 \\ \nabla \cdot \mathbf{h} = 0 \end{array} \right. \quad (3)$$

where

$$\begin{aligned} \sigma^2 &= \nu / \eta \\ \mathcal{P}_r^m &= \eta / k \end{aligned}$$

$$M = H_0 d (\mu_e / \rho_0 \nu \eta)^{1/2} = \text{Hartmann number}$$

$$R_a^m = \frac{\alpha g d^3 (T_0 - T_d)}{\eta k} = \text{magnetic Rayleigh number}$$

ber

$$\beta_H = \beta_1 H_0 / \eta = \text{Hall number}$$

$$\beta_I = \beta_2 H_0^2 / \eta = \text{ion-slip number.}$$

We notice that $R_a^m = \sigma^2 R_a$, where R_a is the hydrodynamic Rayleigh number.

Methods

On introducing the vorticity and the electrical current density fields

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \quad , \quad \mathbf{j} = \nabla \times \mathbf{h},$$

on taking the *curl* and the *curl curl* of (3)₁, the *curl* of the second equation (3)₂, and then the third components of the obtained equations, it turns out that

$$\left\{ \begin{array}{l} \frac{\partial h_3}{\partial t} = \partial_3 u_3 - \beta_H \partial_3 j_3 + (1 + \beta_I) \Delta_2 h_3 \\ \frac{\partial \omega_3}{\partial t} = \sigma^2 M^2 \partial_3 j_3 + \sigma^2 \Delta_2 \omega_3 \\ \frac{\partial j_3}{\partial t} = \partial_3 \omega_3 + \Delta_2^{(1)} j_3 + \beta_H \partial_3 \Delta_2 h_3 + \\ \quad + (1 + \beta_I) \partial_{33}^2 j_3 \\ \frac{\partial}{\partial t} (\Delta_2 u_3) = \sigma^2 M^2 \partial_3 \Delta_2 h_3 + \sigma^2 \Delta_2 \Delta_2 u_3 + \\ \quad + R_a^m \Delta_2^{(1)} \theta \\ \mathcal{P}_r^m \frac{\partial \theta}{\partial t} = u_3 + \Delta_2 \theta \end{array} \right. \quad (4)$$

where Δ_2 is the Laplace operator, $\Delta_2^{(1)} = \partial_{11}^2 + \partial_{22}^2$ is the plane Laplace operator and, as usual, differentiations with respect to the third variable are denoted with ∂_3 and ∂_{33}^2 .

To the equations (3) we append the usual boundary conditions of the "stress free" case [5]

$$u_3 = \theta = j_3 = h_3 = \partial_3 \omega_3 = \partial_{33}^2 u_3 = \Delta_2 u_3 = \Delta_2 \theta = 0 \quad \text{on } x_3 = 0, 1. \quad (5)$$

By the analysis into normal mode, we study the onset of instability in the critical state and we investigate on the critical Rayleigh curves and their properties for the stability.

Moreover we perform the linear stability analysis via the Liapunov direct method, to obtain the critical Rayleigh number, by solving the Euler-Lagrange equations related with variational maximum problem.

Results

We prove that, the conditions ensuring the validity of the P.E.S. in the anisotropic MHD Rayleigh-Bénard problem are given by [6]:

$$\left\{ \begin{array}{l} \frac{\eta}{\kappa} > 1 \\ \frac{\beta_I}{\beta_H} > 1 \end{array} \right. \quad (6)$$

Moreover, under the conditions (6), we prove that, in the "free-free" case, for high levels of the anisotropic currents, according with (6)₂, the critical Rayleigh curves of our anisotropic problem approach the critical Rayleigh curve of the classical hydrodynamic case and the anisotropic Rayleigh critical number is exactly the same of the hydrodynamic one, namely (see Fig. 1, 2):

$$R_c^{(H,I)} = R_a^{(c)} = \frac{27}{4} \pi^4 \quad (7)$$

Conclusions

The conditions (6)₂ clearly show that the validity of the P.E.S. (and therefore the onset of instability as steady convection), as well as the stability, in the anisotropic MHD Reyleigh-Bénard problem, at least in the linear case, can be assured also with strong values of β_H and β_I , namely:

"high levels of the electrical anisotropic currents can assure stability as well".

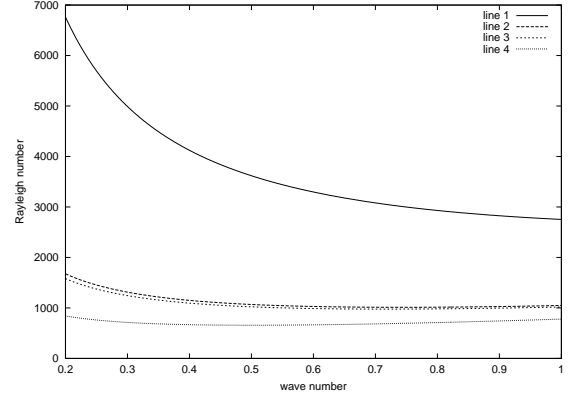


Figure 1: Critical Rayleigh curves in the HD case (bottom), in the isotropic MHD case (top) with $M = 10$, and in the anisotropic one with $\beta_H = 4$, $\beta_I = 5, 6$ (in the middle).

REMARK. - We note that the *nonlinear stability case* for the anisotropic MHD Bénard problem has been investigated in the paper [8], in which it has been

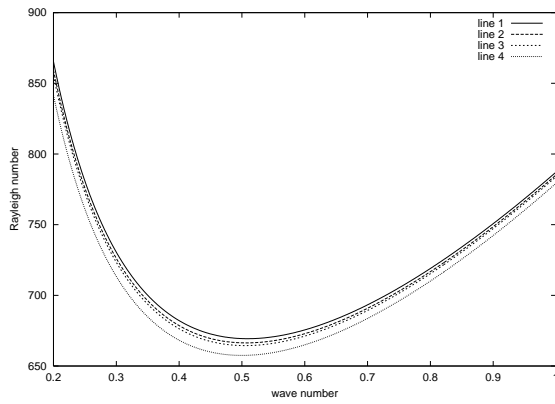


Figure 2: Critical Rayleigh curves in the HD case (bottom) and with $M = 10$, $\beta_H = 100$, $\beta_I = 200, 300, 400$ (from the top).

proved that asymptotic energy stability is assured by the conditions

$$\begin{cases} \beta_I < 1 \\ \forall \beta_H > 0 \end{cases}$$

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