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Robust heteroclinic cycles in two-dimensional Rayleigh-Bénard convection without Boussinesq symmetry

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Abstract

In two-dimensional convection midplane reflection symmetry is absent whenever the layer is heated using constant electrical power, or heat is lost from the top boundary. Near the 1:2 spatial resonance in low Prandtl number fluids these effects open up intervals of Rayleigh number with no stable solutions in the form of steady convection or steadily traveling waves. Direct numerical simulations in one such region show that the dynamics take the form of a nearly heteroclinic modulated traveling wave.

Introduction

In this paper we explore further the consequences of breaking the midplane reflection symmetry (sometimes called the Boussinesq symmetry) that is present in most models of convection. This symmetry is broken when the Boussinesq approximation becomes inaccurate and the depth dependence of the fluid parameters must be included in the theory, or when the boundary conditions at the top and bottom of the layer differ. In a typical experiment both effects are present, and have similar consequences. In the present paper we therefore focus on a Boussinesq fluid but vary the boundary conditions at the top. In earlier work [1] we showed that the presence of Boussinesq symmetry may have dramatic consequences even in two-dimensional convection, because of its effect on (some) mode interaction points, and used numerical branch following techniques to explore the resulting behavior farther from these points. At a mode interaction point the conduction state of the system is marginally stable with respect to two spatial modes with distinct wavenumbers, of the form $2\pi n/L$, where L is the imposed spatial period in the horizontal. In systems without the Boussinesq symmetry the n_1:n_2=1:2 resonance is the dominant resonance, but its effects are suppressed when this symmetry is present, with the 1:3 resonance taking its place. Since the dynamics near these two resonances are quite different [2, 3] it is of interest to explore the crossover from one type of behavior to the other, particularly in low Prandtl number fluids. In earlier papers [4, 5] we explored the consequences of homotopically continuing the velocity boundary conditions at the top from no-slip to free-slip, while keeping the lower boundary no-slip, and discovered that the progressive loss of symmetry opens up intervals in Rayleigh number in which none of the simple solutions known to be created near the 1:2 mode interaction (the two pure modes with $n = 1$ and $n = 2$, various types of mixed steady states, and traveling waves) are stable. The results are similar if the velocity boundary conditions at top and bottom remain no-slip but the top layer radiates heat to the outside via Newton's law of cooling, while the bottom temperature remains fixed [6]. In this paper we report the results of direct numerical simulations of the governing equations in one of these interesting regions.

Methods

In this section we summarize the equations describing the system shown in fig. 1 together with their basic properties. The figure shows a layer of fluid of depth d confined between two boundaries in contact with heat baths.

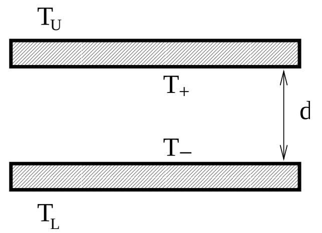

Figure 1: Sketch of the fluid layer.

We model the boundary conditions on the fluid using Biot numbers $B_\pm$:

$$\begin{aligned} \frac{dT_-}{dz} &= -\frac{B_-}{d}(T_L - T_-) \quad \text{at} \quad z = -d/2, \\ \frac{dT_+}{dz} &= -\frac{B_+}{d}(T_+ - T_U) \quad \text{at} \quad z = d/2. \end{aligned}$$

Here T_L, T_U are the temperatures of the lower (L) and upper (U) heat baths, and T_-, T_+ are the temperatures in the fluid right next to the lower and upper boundaries. Thus conducting boundaries correspond to $B = \infty$ while an insulating boundary corresponds to $B = 0$. Note that, by hypothesis, T_L, T_U are constants independent of time, while T_-, T_+ fluctuate in response to the motion of the fluid, allowing the temperature difference $\Delta T \equiv T_- - T_+$ across the fluid to change as the system evolves. However, the (modified) Rayleigh number $Ra' \equiv \alpha g d^3 \Delta T^c/\kappa\nu$, where

$$\Delta T^c \equiv \frac{B_+ B_- \Delta T - d(B_+ \partial_z T_- + B_- \partial_z T_+)}{B_+ B_- + B_+ + B_-},$$

is independent of time [6] and hence constitutes to a suitable bifurcation parameter for the system. The equations are written in terms of a mean flow $\mathbf{U} = (U, 0)$ and its fluctuating part $\mathbf{v}' = (-\partial_z \chi', \partial_x \chi')$, where $\overline{\mathbf{v}}' = \overline{\chi'} = 0$, with the overline indicating an average over the horizontal period, and take the (dimensionless) form

$$\begin{aligned} (\partial_t - \sigma \partial_{zz}^2) U + \partial_z \overline{v_x' v_z'} &= 0 \quad (1a) \\ (\partial_t + U\partial_x - \sigma\nabla^2)\omega' + Ra'\, \sigma \partial_x \theta & \\ + \partial_{zz}^2 U \partial_x \chi' + \frac{\partial(\chi', \omega')}{\partial(x,z)} &= 0 \quad (1b) \\ (\partial_t + U\partial_x - \nabla^2)\theta - \partial_x \chi' + \frac{\partial(\chi', \theta)}{\partial(x,z)} &= 0 \quad (1c) \end{aligned}$$

where $\omega' = -\nabla^2 \chi'$ and σ is the Prandtl number. The boundary conditions on the temperature fluctuation θ, measured in units of ΔT^c, are

$$(1 - B_\mp^*)\theta_z = \pm B_\mp^* \theta \quad \text{at} \quad z = \mp 1/2, \qquad (1d)$$

where $B_\pm^* = B_\pm/(1 + B_\pm)$. The modified Biot numbers $B_\pm^*$ are convenient for numerical exploration, and are such that $B = 0(\infty)$ corresponds to $B^* = 0(1)$. For no-slip boundaries the velocity boundary conditions are

$$U = \chi' = \partial_z \chi' = 0 \quad \text{at} \quad z = \pm 1/2. \qquad (1e)$$

The Boussinesq symmetry is therefore broken when $B_+^* \neq B_-^*$.

Results

Fig. 2 shows the Nusselt number N as a function of Ra' obtained from equations (1a-e) for $\sigma = 0.1$, $B_+^* = 0.8$, $B_-^* = 1$ and $k = 2.08$. For these parameters the 1:2 interaction occurs at $Ra' = 1786.03$, $k = 2.04$. The figure shows that the branch of steady states with $n = 1$ is the first to bifurcate from the conduction state as Ra' increases, followed the pure modes $n = 2$. The $n = 1$ branch is initially stable but loses stability at a parity-breaking bifurcation at $Ra' \approx 1826$ that produces a pair of traveling waves (dashed line), one traveling to the left and the other to the right. These traveling waves (hereafter TW) bifurcate supercritically and hence are initially stable.

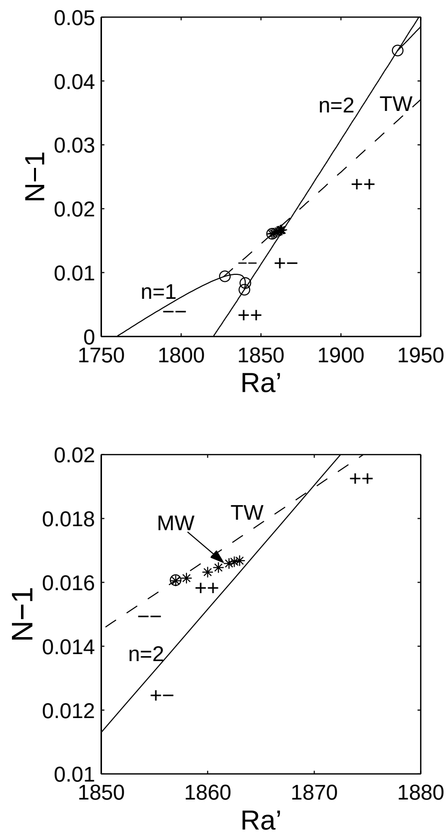

Figure 2: Time-averaged Nusselt number N as a function of Ra' for $\sigma = 0.1$, $B_+^* = 0.8$, $B_-^* = 1$ and $k = 2.08$, showing TW (- - -) and MW (*). Local bifurcations are denoted by open circles.

The (now unstable) $n = 1$ branch undergoes a saddle-node bifurcation ($Ra' = 1840.1$) before terminating on the $n = 2$ branch in a steady state bifurcation ($Ra' = 1839.6$). However, the $n = 2$ states remain unstable until a (subcritical) steady state bifurcation at $Ra' \approx 1935$ that gives rise to a mixed mode. Thus there are no stable steady states in the interval $1826 < Ra' < 1935$. Since the TW themselves lose stability at a tertiary Hopf bifurcation producing modulated traveling waves (MW) at $Ra' \approx 1856$ the dynamics in $1856 < Ra' < 1936$ must have nontrivial time-dependence. In the remainder of this paper we describe some of the solutions we have located in this interval using direct numerical simulation of equations (1a-e).

Fig. 3 shows the change in the MW as Ra' increases beyond $Ra' = 1856$ in terms of $\mathrm{Re}\,\theta_2(t)$, where θ_2 is the $n = 2$ Fourier amplitude of the temperature θ. The figure shows that the modulation period increases dramatically near $Ra' = 1863$, suggesting the presence of a global bifurcation.

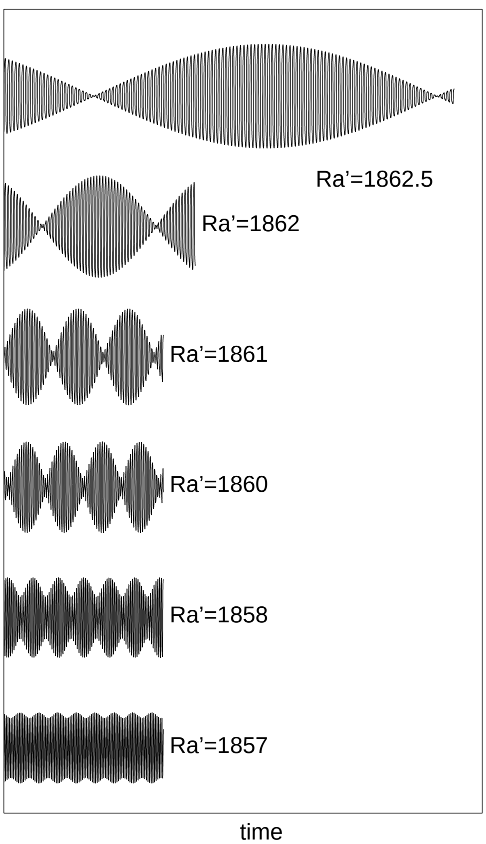

Figure 3: Time series $\mathrm{Re}\,\theta_2(t)$ for several values of Ra' near the global bifurcation at $Ra' \approx 1863$.

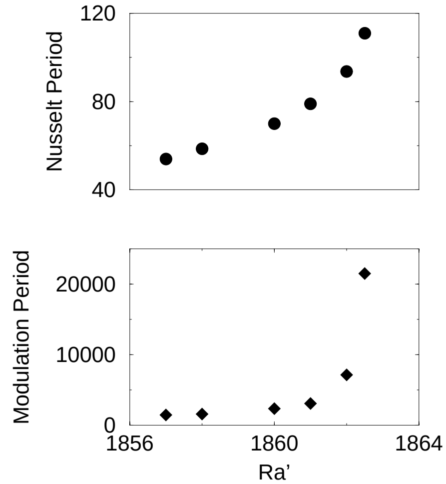

Figure 4: (a) The Nusselt number period and (b) the modulation period from fig. 3 as functions of Ra'.

Fig. 4 shows the modulation period as well as the Nusselt number period as functions of Ra'. The simulations indicate that the abrupt increase in the modulation period is a consequence of the approach of the phase space trajectory closer and closer to reflection-invariant subspaces. As this occurs the Eulerian mean flow associated with the MW disappears. Beyond $Ra' \approx 1863$ the solution becomes trapped in successive reflection-invariant subspaces, and appears to take the form of an attracting structurally stable heteroclinic cycle of the type present in the normal form for the 1:2 resonance [7, 8]. These reflection-invariant subspaces contain the $n = 2$ pure mode, indicating that the torus is colliding with the (circle of) of pure modes, cf. [7, 8]. The structurally stable heteroclinic cycle that results from this interaction connects an $n = 2$ steady state with its translate by $L/4$. As shown in [9] the stability of such a cycle is determined by the eigenvalues of this state; computation of these eigenvalues confirms that the cycle is attracting for $Ra' > 1863$. However, the resulting theory suggests that the period of the oscillations should lengthen with time as the trajectory approaches closer and closer to the heteroclinic cycle. In fact in our simulations we find that the period saturates, and we (may) obtain a periodic cycle (fig. 5a). In this plot the heavy line shows $\mathrm{Re}\,\theta_2$; we see that the system spends extended periods of time in a pure $n = 2$ state. The thin lines show the real (solid) and imaginary (dashed) parts of θ_1; these have twice the period of θ_2, as expected of a TW, and are only excited during the brief transitions from an $n = 2$ state to its $L/4$ translate. Identical time series were obtained with two different codes, provided both were started from identical initial conditions, suggesting that the periodicity of the time series is not an artifact of numerical discretization or round-off errors. Indeed, this time series is consistent with the constraint $\arg\theta_2 = 2\arg\theta_1$ (mod π) corresponding to an invariant subspace of the system: a jump in $\arg\theta_2$ by π (i.e., a translation by $L/4$) corresponds to a jump in $\arg\theta_1$ of $\pi/2$ etc., as in fig. 5a. This conclusion is consistent with the numerical observation that periodic switching occurs when integration is started from an $n = 1$ state with $\theta_1(t = 0)$ either purely real or purely imaginary. In contrast, if one starts with a general choice of $\theta_1(t = 0)$ the switching eventually stops being periodic and becomes irregular (fig. 5b). The characteristic time T between successive switches also depends on initial conditions. Similar results obtain for $Ra' = 1870$, albeit with larger values of T.

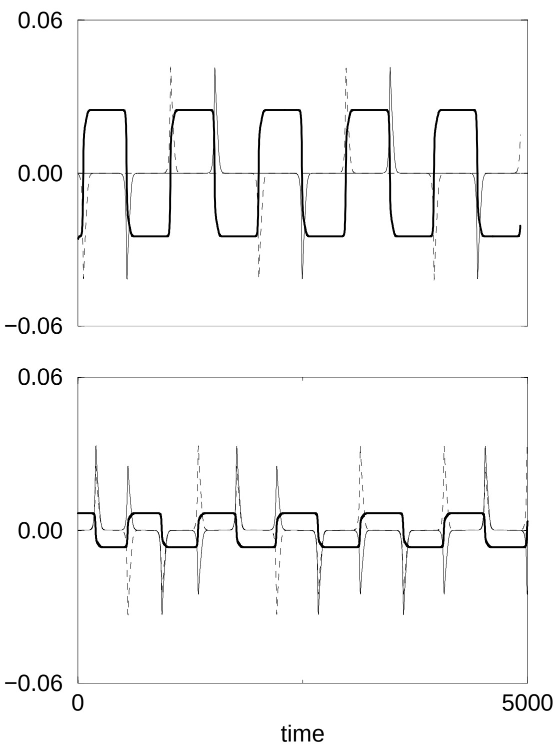

Figure 5: Time series Reθ_1 (—), Imθ_1 (- - -), and Reθ_2 (heavy line) for two different initial conditions when $Ra' = 1865$. In (a) the cycle is periodic with $T \approx 490$; in (b) it is irregular with $T \sim 460$.

Acknowledgements

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