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THE COMPLETE LINEAR STABILITY BOUNDARY FOR SPIRAL POISEUILLE FLOW

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Abstract

The stability boundary for spiral Poiseuille flow determined by Takeuchi and Jankowski (*J. Fluid Mech.* **102**, 101, 1981) has been extended to the entire range of Reynolds numbers for which the flow is stable. For $\eta = R_i/R_o = 0.5$, the results are presented in terms of a critical Taylor number that depends on Re. Beyond the local maximum reported by Takeuchi and Jankowski, the critical Ta passes through a broad plateau before precipitously decreasing to zero at the critical Re for annular Poiseuille flow. The critical azimuthal wavenumber also decreases from $m = 5$ at Takeuchi and Jankowski's highest Re (100) to $m = 2$ at the nonrotating critical Re. At $Ta = 0$, the critical Re and azimuthal wavenumber agree well with values of Sadeghi and Higgins (*Phys. Fluids A*, **3**, 2092, 1991).

Introduction

The stability of spiral Poiseuille flow [1], in which an axial pressure gradient is simultaneously applied, has been investigated by Takeuchi and Jankowski [2] and Ng and Turner [3]. A comprehensive summary of previous work was provided by Takeuchi and Jankowski [2].

Takeuchi and Jankowski [2] experimentally and computationally investigated the case with the inner cylinder (radius R_i) rotating at constant angular velocity $\Omega_i > 0$, and the outer cylinder (radius R_o) fixed ($\Omega_o = 0$), co-rotating ($\Omega_o > 0$), or counter-rotating ($\Omega_o < 0$) in a "wide" gap case (i.e., $\eta = R_i/R_o = 0.5$). For $\mu = \Omega_o/\Omega_i = -0.5$, 0, and 0.2, they found in computations over the range $0 \le Re \le 100$ and in experiments over the range $0 \le Re \le 150$, where $Re = \bar{V}_{ZB}(R_o - R_i)/\nu$ and $\bar{V}_{ZB}$ is the mean axial velocity of the base flow, that the critical Taylor number Ta_{crit} (where $Ta = \Omega_i (R_o - R_i)^2/\nu$) is strongly affected by the axial flow and the angular velocity ratio. Their results for nonnegative μ show that as Re increases from zero (i.e., no axial flow), Ta_{crit} increases (i.e., the flow is stabilized). This continues until a (globally) maximum value of Ta is reached, beyond which axial flow destabilizes the base flow, although for $\mu = 0$ Ta_{crit} still exceeds its value at $Re = 0$ over the entire range considered. For negative μ, Ta_{crit} decreases (i.e., the flow is destabilized) as Re increases from zero, and continues to decrease until a global minimum is reached, beyond which axial flow is stabilizing. For $\mu = 0$ and 0.2, their experimental and computational results are in excellent agreement for $Re < 40$, with the experimental values of Ta_{crit} beyond that point exceeding the computed values to an extent that increases with Re. For $\mu = -0.5$, the experimental Ta_{crit} exceeds the computed value over the entire range of Re. For each μ, however, the overall characters of the experimental and computed stability boundaries were similar.

In contemporaneous work, Ng and Turner [3] computationally investigated the same class of flows, but considered Re up to 6000. In their work the outer cylinder is stationary (i.e., $\mu = 0$) and the radius ratios considered are $\eta = 0.77$ and 0.95. Their results show that as Re increases from zero, Ta_c increases. Instead of reaching a maximum and then decreasing as Re is increased further (as in the $\eta = 0.5$ case), Ta_{crit} reaches a broad plateau. This behavior was found for both values of η investigated.

We know from the work of Mott and Joseph [4] and Sadeghi and Higgins [5] that annular Poiseuille flow becomes linearly unstable at a finite Re_{crit} for $Ta = 0$. From this, we know that $Ta_{crit}(Re)$ must ultimately decrease to zero as $Re \to \mathrm{Re}_{crit}(Ta = 0)$. The nature of the stability boundary near $\mathrm{Re}_{crit}(Ta = 0)$ is not elucidated by the previous work, which stopped at Re below the nonrotating critical Re.

In the present work, we complete the stability boundary for the cases considered by Takeuchi and Jankowski [2] and Ng and Turner [3]. We also extend the computations to the "small-η" case ($\eta = 0.1$), as distinguished from the "large-gap" or "wide-gap" $\eta = 0.5$ case.

Numerical Methods

We use Chebyshev collocation to discretize the disturbance equations. This leads to a generalized matrix eigenvalue problem, which is solved using the numerical library LAPACK.

For a given set of parameters (Re, μ, and η), we seek the maximum Taylor number for which the real part of each growth rate does not exceed zero for all real k and integer m. At this critical Ta, it can be shown that the eigenvalues corresponding to $k<0$ are complex conjugates of those for $k>0$ at each m. Thus, we restrict k to be positive and allow m to take on all integer values.

For given Re, μ, and η, we locate two values of Ta for each m: one stable for all k considered in the range $0 \le k \le 100$, and one unstable for at least one value of k We then do an "axial wavenumber traverse," computing the maximum value of the real part of the growth rate over the wavenumber range for each Ta For each value of Ta, one selects a wavenumber on each side of the maximum growth rate and fits a quadratic polynomial to the three pairs (Ta, σ_r). From this, one iterates on the axial wavenumber k until the real part of the growth rate converges. Once the maximum value of growth rate and the corresponding value of k have been found for each of the two values of Ta, a secant iteration on Ta is used to determine Ta_{crit},. At this point, one has critical values of Ta and k for a given value of m. The above procedure is repeated for $-M \le m \le M$ to find the global minimum, where M is specified for each value of η below.

To validate the code, we have compared our results to previous results for spiral Poiseuille flow [2-3], and obtained excellent agreement between our calculations and theirs. We then compared our results to previous annular Poiseuille computations [5]. Again, there is excellent agreement among computed values of Ta_{crit} (less than one percent difference for $\eta = 0.5$ and $\mu = 0$).

To insure that the critical values are independent of resolution, we performed convergence tests for a range of Re. In general, the resolution needed to ensure a specified degree of accuracy increases with decreasing η, while for a given η and Ta, the required resolution increases with increasing Re.

Results

We consider three values of μ, the ratio of rotation rates of the outer to the inner cylinders. The first case, $\mu = 0$, corresponds to a fixed outer cylinder, while the second ($\mu = 0.2$) and third ($\mu = -0.5$) cases pertain to co-rotating and counter-rotating cylinders, respectively. We consider three radius ratios, $\eta = 0.5$, 0.77, and 0.95, with progressively smaller annular gaps. For the two wider gaps, $\eta = 0.5$, 0.77, computations were performed at all three values of μ. For $\eta = 0.95$, only the $\mu = 0$ case was investigated.

Fixed outer cylinder ($\mu = 0$)

Stability boundaries were determined for $\eta = 0.5$, 0.77, and 0.95. For $\eta = 0.5$, Takeuchi and Jankowski [2] determined stability boundaries computationally for $0 \le Re \le 100$ and experimentally for $0 \le Re \le 150$. Annular Poiseuille flow has been shown (in the present work) to be stable up to $Re = 10359$. For $\eta = 0.95$, Ng and Turner [3] considered the stability boundary with respect to axisymmetric disturbances up to $Re = 7739.5$, at which point the nonrotating flow becomes unstable. For $\eta = 0.77$ and 0.95, they computed the stability boundary for nonaxisymmetric disturbances up to $Re = 6000$. Thus, for $\mu = 0$, the stability boundary remains undetermined for $150 \le Re \le 10359$ at $\eta = 0.5$, for $6000 \le Re \le 8883.3$ at $\eta = 0.77$ (with the upper limit of stable flow being determined here), and $6000 \le Re \le 7739.5$ at $\eta = 0.95$. In no case has the connection to the $Ta = 0$ Tollmien-Schlichting instability been established.

In all cases, we complete the stability boundary, with respect to axisymmetric and nonaxisymmetric disturbances of infinitesimal magnitude, by extending the range of Re to the value at which $Ta_{crit} = 0$ (i.e., the Re at which annular Poiseuille flow becomes linearly unstable).

For $\eta = 0.5$, the results show that as Re approaches zero, Ta_{crit} approaches 68.19 [6], the value corresponding to the onset of Taylor vortices absent mean axial flow. Beyond some critical Re (10359), there is no range of stable Ta. This critical Re, for which the nonrotating flow is marginally stable, is the critical Re for annular Poiseuille flow. As mentioned above, the maximum values of Re considered by Takeuchi and Jankowski [2] in their computations and experiments were only 100 and 150, respectively. Here, we complete the stability boundary by extending the Re range up to $Re_{crit} = 10359$.

Several interesting trends are found over this range of Re ($0 \le Re \le 10359$). The scalloped nature of the curve, corresponding to discrete changes in the critical azimuthal wavenumber (m_{crit}), is apparent. Further, for $Re < 10$ and $400 \le Re \le 10000$, Ta_{crit} remains almost constant. This indicates that the axial shear flow instability plays no significant role in these ranges of Re. However, in the high Reynolds number range ($400 \le Re \le 10000$) the nearly constant value of Ta_{crit} (≈ 88) is greater than in the lower Re range ($Ta_{crit} \approx 68.19$ for $Re < 10$), indicating that the axial flow has a more stabilizing effect at higher Re.

For $10 < Re \le 60$, the flow becomes increasingly stable as Re increases and reaches a maximum at $Re \approx 60$. Beyond this, there is a monotonic decrease in Ta_{crit} up to $Re \approx 400$, indicating a destabilization of the flow in this range of Re. At very high Reynolds numbers, we see a precipitous fall in Ta_{crit}. This corresponds to a transition from centrifugal instability to Tollmien-Schlichting instability.

For $\eta = 0.77$, Ta_{crit} approaches 26.57 as Re approaches zero. For this η, Ng and Turner [3] computed the stability boundary with respect to nonaxisymmetric disturbances for Re up to 6000. We find that $Ta_{crit} = 0$ occurs at $Re_{crit} = 8883.3$, and complete the stability boundary by computing Ta_{crit} for Re up to 8883.3.

There are qualitative as well as quantitative differences between the stability boundaries for $\eta = 0.5$ and 0.77. The scalloped behavior for $\eta = 0.77$ is less pronounced than in the $\eta = 0.5$ case. We find that Ta_{crit} increases with increasing

Re up to $Re = 200$. This behavior contrasts with $\eta = 0.5$, where we see both stabilization and destabilization of the flow for $10 \le Re \le 400$. Further, Ta_{crit} for $\eta = 0.77$ remains nearly constant in the ranges $Re < 10$ and $200 \le Re \le 8000$ Qualitatively similar behavior was observed for $\eta = 0.5$ For $\eta = 0.77$, the nearly constant value of Ta_{crit} (≈ 59) is greater in the $200 < Re \le 8000$ range than in the $Re < 10$ range ($Ta \approx 26$), indicating that the flow is more stable at higher Re, as for $\eta = 0.5$. At very high values of Re, we again see a precipitous drop in Ta_{crit}, similar to that of $\eta = 0.5$. The values of Ta_{crit} for $\eta = 0.5$ are higher than those for $\eta = 0.77$ for all Re considered.

For $\eta = 0.95$, Ng and Turner [3] computed Ta_{crit} for Re up to 6000 allowing for nonaxisymmetric disturbances, and up to 7739.5 allowing for axisymmetric disturbances. We have completed the stability boundary by extending the range of Reynolds numbers considered up to $Re_{crit} = 7739.5$, at which annular Poiseuille flow becomes linearly unstable to an axisymmetric disturbance.

Qualitatively, the stability boundary is similar to that for $\eta = 0.77$. For this case, Ta_{crit} remains nearly constant ($Ta \approx 9$) for $Re < 10$ and for $1000 < Re \le 7000$, with Ta_{crit} ($Ta \approx 59$) being greater in the high Re range. The values of Ta_{crit} for $\eta = 0.95$ are less than both $\eta = 0.5$ and $\eta = 0.77$.

Co-rotating outer cylinder ($\mu > 0$)

For $\mu = 0.2$, we then consider the stability of the flow for $\eta = 0.5$ and 0.77. For $\eta = 0.5$, we complete the stability boundary determined by Takeuchi and Jankowski [2]. They determined the stability boundary computationally for $0 \le Re \le 100$ and experimentally for $0 \le Re \le 150$. We extend the values of Re considered up to $Re_{crit} = 10359$ for which the annular Poiseuille flow is marginally stable.

For $\eta = 0.5$, the complete stability boundary is presented in Figure 5, which shows that as Re approaches zero, Ta_{crit} approaches 124.72, the value corresponding to the onset of Taylor vortices absent mean axial flow. Beyond $Re_{crit} = 10359$, there is no range of stable Ta.

The trends in the stability boundary for the intermediate values of Re ($0 \le Re \le 10359$) are as follows. The scalloped nature of the curve, corresponding to discrete changes in the critical azimuthal wavenumber (m_{crit}), is evident from figure 5. The critical Ta remains almost constant for $Re < 7$ and $600 \le Re \le 10000$. However, the nearly constant value of Ta_{crit} (≈ 124.72) is greater in the low Reynolds number range, indicating that shear destabilizes the flow against centrifugal instability for $600 \le Re \le 10000$. This trend is opposite to what we have seen for $\eta = 0.5$ and $\mu = 0$, where the asymptotic value of Ta_{crit} was higher in the high Re range. In the intermediate range of Re, there is a stabilization of the flow up to $Re \approx 38$ (where Ta_{crit} attains a maximum) due to increasing Ta_{crit} and a destabilization in the range $38 < Re < 600$. The precipitous drop in Ta_{crit} at high Re indicates that instabilities of Tollmien-Schlichting type play a dominant role here.

For $\eta = 0.77$, Ta_{crit} approaches 28.897 as Re approaches zero. In this case, we determined the stability boundary by computing Ta_{crit} for Re up to $Re_{crit} = 8883.3$.

The scalloping behavior is not as pronounced as in the $\eta = 0.5$ case. In general, Ta_{crit} increases with Re, except at high Re where we again see a precipitous drop in Ta_{crit} This behavior contrasts with $\eta = 0.5$, where we see both stabilization and destabilization of the flow. The critical Ta for $\eta = 0.77$ remains almost constant in the ranges $Re < 10$ and $200 \le Re \le 8000$. The stability boundary for $\eta = 0.5$ also exhibits a similar behavior. For $\eta = 0.77$ the nearly constant value of Ta_{crit} is greater (≈ 56) in the $200 \le Re \le 8000$ range than in the $Re < 10$ range (≈ 30), indicating that the flow is more stable in the higher Re range. At very high values of Re we see a precipitous drop in Ta_{crit}, similar to previous cases. The values of Ta_{crit} for $\eta = 0.5$ are higher than those for $\eta = 0.77$ for all values of Re considered.

Counter-rotating outer cylinder ($\mu < 0$)

For $\mu = -0.5$, stability boundaries were determined for $\eta = 0.5$ and 0.77. For $\eta = 0.5$, we complete the stability boundary partially determined by Takeuchi and Jankowski [2]. We extend the values of Re considered up to $Re_{crit} = 10359$, at which point the nonrotating annular Poiseuille flow is marginally stable.

For $\eta = 0.5$, our results show that as Re approaches zero, Ta_{crit} approaches 111.31, the value corresponding to the onset of Taylor vortices absent mean axial flow. Beyond $Re_{crit} = 10359$, there is no range of stable Ta.

The trends in the stability boundary for intermediate values $0 < Re < 10359$ are as follows. The scalloped nature of the curve, corresponding to discrete changes in the m_{crit}, is evident from figure 7. The critical Ta remains almost constant only for $1000 < Re \le 10000$ with no such asymptotic behavior seen in the low Re range. There is a destabilization of the flow, due to decreasing Ta_{crit}, for $0 < Re \le 10$, followed by a stabilization up to $Re \approx 200$. There is a marginal decrease in Ta_{crit} for $200 < Re \le 1000$ Finally, at very high Re ($Re > 10000$), we see the precipitous drop in Ta_{crit}.

For $\eta = 0.77$, Ta_{crit} approaches the limiting value 32.826 as Re approaches zero. We have determined the complete stability boundary by computing Ta_{crit} for Re up to $Re_{crit} = 8883.3$.

There are important differences in the stability boundaries for $\eta = 0.5$ and 0.77. The scalloping behavior for $\eta = 0.77$ is not as pronounced as in the $\eta = 0.5$ case. In general, there is a stabilization of the flow with increasing Re, except at very high Re where there is a precipitous drop. This behavior contrasts with $\eta = 0.5$ where we see destabilization and stabilization of the flow for $0 < Re \leq 1000$. The critical Ta for $\eta = 0.77$ remains almost constant in the ranges $Re < 10$ and $1000 < Re \leq 8000$, in contrast to the stability boundary for $\eta = 0.5$, which is flat in only one range of Re ($1000 < Re \leq 10000$). For $\eta = 0.77$ the nearly constant value of Ta_{crit} (≈ 69) is greater in the $1000 < Re \leq 8000$ range than in the $Re < 10$ range (≈ 33), indicating that the flow is more stable in the higher Re range. At very high values of Re we see a precipitous drop in Ta_{crit}, similar to the $\eta = 0.5$ case.

Acknowledgments

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