STABILITY OF AN INITIALLY, STABLY STRATIFIED FLUID SUBJECTED TO A STEP CHANGE IN TEMPERATURE

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Abstract

The onset of convective instability in an initially quiescent, stably stratified fluid layer between the two horizontal plates is analysed with linear theory. The layer is heated suddenly from below, subjected to a step change in surface temperature. The dimensionless critical time τ_c is to mark the onset of Rayleigh-Bénard convection is obtained numerically by using propagation theory. The results show that disturbances manifest themselves around $\tau = 4\tau_c$ in comparison with available experimental data.

Introduction

When a fluid layer confined between the two horizontal plates is heated rapidly from below, Rayleigh-Bénard convection can set in at a certain time due to buoyancy forces. In this transient-heating system the important problem is to find the critical time to mark the onset of convective motion.

The system considered here is sketched in Fig. 1. Initially the quiescent fluid layer of depth d is stratified stably with temperature $T = T_i$ at the vertical distance $Z = 0$ and $T = T_u$ ($\geq T_i$). Starting from time $t = 0$, the bottom boundary is heated uniformly at a higher temperature T_b. For small time the base temperature profile of heat conduction will be nonlinear and time-dependent. The important parameters in this thermally developing system are the Rayleigh number Ra ($= g\beta\Delta T d^3/(\nu\alpha)$), the Prandtl number Pr ($=\nu/\alpha$) and the temperature ratio γ ($=(T_u - T_i)/(T_b - T_i)$). Here g, β, ΔT, ν and α denote the gravitational acceleration, the thermal expansivity, the temperature difference across the boundaries ($= T_b - T_u$), the kinematic viscosity, and the thermal diffusivity, respectively. The object of this study is to find the dimensionless critical time τ_c to mark the onset of convective instability for a given Pr, Ra and γ. Here τ ($=\alpha t/d^2$) denotes the Fourier number. We will employ propagation theory, which is based on the assumption that temperature disturbances at $\tau = \tau_c$ would be propagated mainly to the thermal penetration depth of conduction state. The present results will complement Kim et al.'s [1] work.

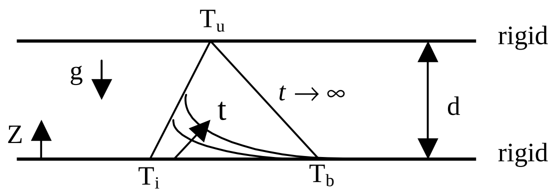

Fig. 1: Temperature profiles in conduction state.

Propagation Theory

For the present system the dimensionless basic temperature θ_0 ($=(T_0 - T_u)/(T_b - T_i)$) of the conduction state can be obtained [1]:

$$\theta_0 = (1-\gamma)(1-z) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi z)}{n}\exp(-n^2\pi^2\tau) \qquad (1)$$

$$\theta_0 = -\gamma(1-\zeta\sqrt{\tau}) + \sum_{n=0}^{\infty}\left\{\operatorname{erfc}\left(\frac{n}{\sqrt{\tau}}+\frac{\zeta}{2}\right) - \operatorname{erfc}\left(\frac{n+1}{\sqrt{\tau}}-\frac{\zeta}{2}\right)\right\} \qquad (2)$$

where $z = Z/d$ and $\zeta = z/\sqrt{\tau}$. Here T_0 denotes the basic temperature. Equations (1) and (2) yield the same temperature profile but have the different coordinates.

Under linear theory the perturbed quantities are expressed in terms of the temperature component θ_1 and the vertical velocity component w_1 as

$$\left(\frac{1}{Pr}\frac{\partial}{\partial\tau} + \nabla^2\right)\nabla^2 w_1 = \nabla_1^2\theta_1 \qquad (3)$$

$$\frac{\partial\theta_1}{\partial\tau} + Ra\, w_1\frac{\partial\theta_0}{\partial z} = \nabla^2\theta_1 \qquad (4)$$

where $\nabla_1^{\ 2}$ denotes the horizontal Laplacian. Here w_1 has the scale of α / d and θ_1 that of $\alpha\nu /(g\beta d^3)$. The proper boundary conditions are given by

$$w_1 = \frac{\partial w_1}{\partial z} = \theta_1 = 0 \quad \text{at} \quad z = 0 \text{ and } 1 \tag{5}$$

which represent no slip and isothermal heating on the boundaries.

For small τ the dimensionless amplitude functions of disturbances are expressed under the normal mode analysis, based on the balance between the viscous and buoyant forces in the z-component of motion [1] :

$$\begin{aligned}&[w_1(\tau, x, y, z), \theta_1(\tau, x, y, z)] \\ &\quad = [\tau w^*(\zeta), \theta^*(\zeta)] \exp[i(a_x x + a_y y)]\end{aligned} \tag{6}$$

where i is the imaginary number, and a_x and a_y denote the wavenumbers. With $\gamma = 0$, substituting Eqs. (6) into (3) and (4) yields

$$\begin{aligned}&\left(D^2 - a^{*2}\right)^2 w^* \\ &\quad = a^{*2}\theta^* - \frac{1}{Pr}\left[\frac{\zeta}{2} D^3 w^* - \frac{\zeta}{2} a^{*2} D w^* + a^{*2} w^*\right]\end{aligned} \tag{7}$$

$$(D^2 - a^{*2})\theta^* = -\frac{\zeta}{2} D\theta^* + Ra^* w^* D\theta_0 \tag{8}$$

where $D = d/d\zeta$, $a^* = \tau^{1/2} a$, $a = \sqrt{a_x^2 + a_y^2}$, and $Ra^* = \tau^{3/2} Ra$. Here a^* and Ra^* are assumed to be eigenvalues. This makes it possible to produce the above self-similar equations including θ_0 $(= \mathrm{erfc}(\zeta / 2))$ from Eq. (2) as a function of ζ $(= z / \sqrt{\tau})$ only, because the upper boundary is replaced by ζ $(= 1/\sqrt{\tau}) \to \infty$ for small τ . Now, for a given *Pr* the minimum Ra^*-value and its corresponding a^*-value are obtained numerically, wherefrom the critical time τ_c and the critical horizontal wavenumber a_c are obtained for a given *Ra*. Also, the critical Rayleigh number Ra_c may be obtained at each τ_c. This means that τ may be fixed as τ_c .

The above procedure is extended to the case of $\gamma > 0$ and also to that of large τ. As shown in Eq. (2), the resulting equations are not self-similar. But we fix τ as τ_c in Eqs. (5), (7) and (8). Now, for a given τ_c , γ and *Pr* the minimum *Ra*-value Ra_c is found. Therefore the propagation theory introduced above is a kind of relaxed frozen-time model.

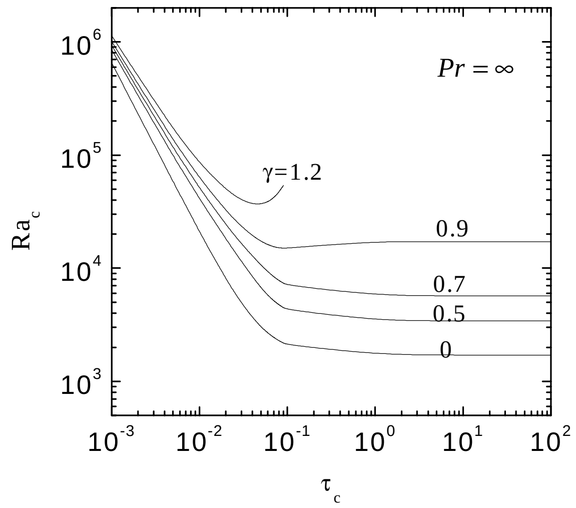

Fig. 2: Effect of the temperature ratio γ on the critical time τ_c .

Results and Discussion

The dimensionless critical time τ_c to mark the onset of instability has been obtained by propagation theory. The system becomes more stable with increasing γ , as shown in Fig. 2, and also as *Pr* decreases. For large τ_c , Ra_c approaches the well-known value of $1708/(1-\gamma)$, independently of *Pr* . For large γ-value both the multiple-cell patterns and the behavior of subcritical state are exhibited, producing the minimum Ra_c-value in the plot of Ra_c vs. τ . For $\gamma \geq 1$ the instabilities will disappear with increasing time and the system becomes unconditionally stable as $\tau \to \infty$. For deep-pool systems of small τ_c the stability criteria are summarized in Kim et al.'s [1] work. Ueda et al. [2] conducted experiments of $Ra =$ 9,000 17,000, $\gamma =$ 0.73 1.67 and $Pr =$ 8,800, and obtained the characteristic time τ_m to mark the detection of manifest convection. Comparison with the present predictions yields the relation of $\tau_m \cong 4\tau_c$ for $\gamma\, Ra^{-1/3} < 0.03$, as shown in Fig. 3. Here their predictions from the amplification theory are also compared. The significant deviation of $4\tau_c$ -values from the last two data points may be caused by a peculiar behavior with $\gamma > 1$ in experiments.

With $\gamma = 0$ the present predictions fit

$$\tau_c = 7.53[1 + (\frac{0.804}{Pr})^{3/4}]^{8/9} Ra^{-2/3} \tag{9}$$

for $\tau_c < 0.01$ with the error bound of 5%. It has been reported [3, 4] that convection would manifest itself at

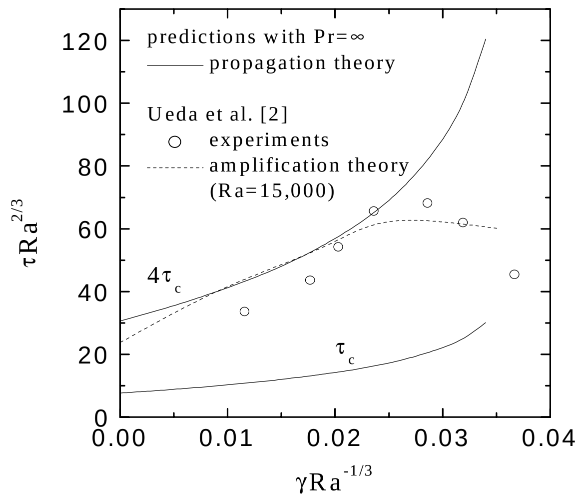

Fig. 3: Comparison of present predictions with Ueda et al.'s [2] results for a given *Ra*.

$\tau = \tau_m$ with the relation of

$$\tau_m \cong 4\tau_c \tag{10}$$

because incipient disturbances at $\tau = \tau_c$ must grow with time. The τ_c -values predicted by the amplification theory [4, 5] and the stochastic model [6] also yield the above relation when τ_c is obtained from Eq. (9). Patrick and Wragg [7] measured the individual mass transfer coefficient with time in electroplating systems, which correspond to those of $Pr > 2{,}000$. Figure 4 shows that their undershoot times are well represented by the relation (10). The undershoot time indicates the characteristic time to exhibit the minimum of the Nusselt number Nu in the plot of Nu vs. τ .

Conclusions

Even though propagation theory is a rather simple model, it seems that the resulting stability criteria are consistent with experimental measurements. The present results show that growing, infinitesimal disturbances set in at $\tau = \tau_c$ and for large-*Pr* systems it grows until detected around $\tau = 4\tau_c$. It is interesting that propagation theory can be applied to the stability analysis of diffusive systems without the loss of generality.

Acknowledgements

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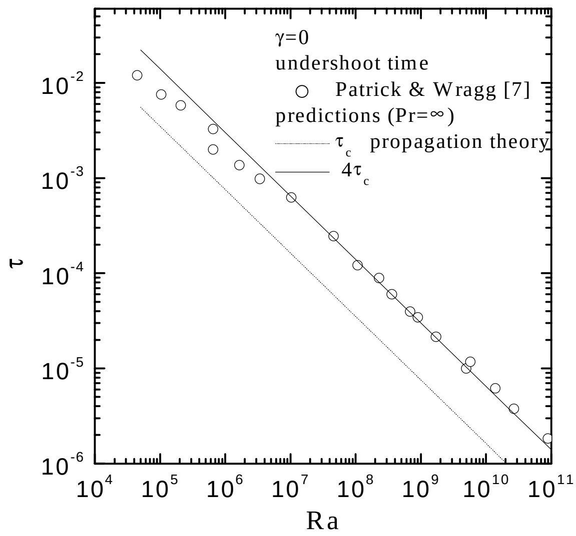

Fig. 4: Comparison of present predictions with experimental data of Patrick and Wragg [7] for a given *Ra*.

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