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Nonnormal Transient Growth in Taylor-Couette Flow.

A. Meseguer *[1] [2]

[1]Oxford University Computing Laboratory; *Wolfson Building, Parks Road, Oxford OX1 3QD, UK*

[2]A. Meseguer; *e-mail: Alvaro.Meseguer@comlab.ox.ac.uk*

Abstract

This work is devoted to the study of transient growth of perturbations in the Taylor-Couette problem due to nonnormal mechanisms. The study is carried out for a particular small gap case and is mostly focused on the linearly stable regime of counter-rotation. The exploration covers a wide range of inner and outer angular speeds as well as axial and azimuthal modes. Clear evidence of transient growth is found as long as the counter-rotation is increased. The numerical results are in agreement with former analyses based on energy methods. Similarities with transient growth mechanisms in plane Couette flow and in Hagen-Poiseuille flow are found. This is reflected in the modulation of the basic circular Couette flow by the presence of azimuthal streaks as a result of the nonmodal growth of initial axisymmetric perturbations. This study might shed some light on the subcritical transition to turbulence which is found experimentally in Taylor-Couette flow when the cylinders rotate in opposite directions.

Introduction

Below the critical values predicted by the linear stability theory, azimuthal Couette flow is stable with respect to infinitesimal perturbations. Nevertheless, experiments carried out by Coles and Van Atta [1, 9] in the 1960's reported new striking phenomena of sudden transition to transient turbulent regimes in the region where the linear theory predicted stability of the basic azimuthal Couette flow. This kind of instability, which Coles termed *catastrophic transition*, cannot be explained by means of eigenvalue analysis of the linearized Navier-Stokes operator, because its spectrum always lies on the stable region of the complex plane. Instead, this subcritical transition seems to be associated with the considerable amplification or transient growth that even very small amplitude perturbations might suffer due to the nonnormality of the linearized operator, i.e. non-orthogonality of its eigenvectors [4]. It has long been known that nonnormality of linearized operators arising in stability analysis of shear flows is responsible for the considerable nonmodal growth of small perturbations [8, 6]. Shear dominated flows such as plane Couette or Hagen-Poiseuille (pipe) flows are linearly stable for all Reynolds numbers although they actually become turbulent due to finite amplitude perturbations which are transiently amplified by nonnormal mechanisms. In [2], a comprehensive exploration of the spectra of the Taylor-Couette eigenvalue problem was provided and the nonnormality of the operator was pointed out. The purpose of this work is to provide evidence of a remarkable energy transient growth of perturbations based on the linear, but nonmodal, stability analysis of the azimuthal Couette flow under those circumstances.

Mathematical formulation

We consider an incompressible fluid of kinematic viscosity ν which is contained between two concentric rotating cylinders whose inner and outer radii and angular velocities are r_i^*, r_o^* and Ω_i, Ω_o respectively. The independent dimensionless parameters appearing in this problem are the radius ratio $\eta = r_i^*/r_o^*$, which fixes the geometry of the annulus, and the Couette flow Reynolds numbers $Ri = dr_i\Omega_i/\nu$ and $Ro = dr_o\Omega_o/\nu$ of the rotating cylinders. Henceforth, all variables will be rendered dimensionless using d, d^2/ν, ν^2/d^2 as units for space, time and the reduced pressure (p^*/ρ^*), respectively.
The Navier–Stokes equation and the incompressibility condition for this scaling take the form

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \Delta\mathbf{v}, \quad \nabla\cdot\mathbf{v} = 0 \qquad (1)$$

Let $\mathbf{v} = v_r\,\mathbf{e}_r + v_\theta\,\mathbf{e}_\theta + v_z\,\mathbf{e}_z = (v_r, v_\theta, v_z)$ be the velocity vector field $\mathbf{v}$ a function of the cylindrical coordinates (r, θ, z) and time t. The basic azimuthal

Couette flow $\mathbf{v}^{\rm B} = (v_r^{\rm B}, v_\theta^{\rm B}, v_z^{\rm B})$ is obtained by assuming independence with respect to t, θ and z:

$$v_r^{\rm B} = 0, \quad v_\theta^{\rm B} = Ar + \frac{B}{r}, \quad v_z^{\rm B} = 0, \tag{2}$$

where $r \in (r_i, r_o)$, $A = (Ro - \eta Ri)/(1+\eta)$, $B = \eta(Ri - \eta Ro)/(1-\eta)(1-\eta^2)$, $r_i = \eta/(1-\eta)$ and $r_o = 1/(1-\eta)$. The basic flow is perturbed by a small disturbance which is assumed to be periodic in the azimuthal and axial coordinates:

$$\mathbf{v}(r,\theta,z,t) = \mathbf{v}^{\rm B} + \mathbf{u}(r){\rm e}^{{\rm i}(n\theta+kz)+\lambda t}, \tag{3}$$
$$p(r,\theta,z,t) = p^{\rm B} + q(r){\rm e}^{{\rm i}(n\theta+kz)+\lambda t}, \tag{4}$$

where $n \in \mathbb{Z}$, $k \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. In addition, the perturbation of the velocity field must satisfy the solenoidal condition

$$\nabla \cdot [{\rm e}^{{\rm i}(n\theta+kz)}\mathbf{u}(r)] = 0, \tag{5}$$

where $\mathbf{u} = (u_r, u_\theta, u_z)$ must cancel at the radial boundaries

$$\mathbf{u}(r_i) = \mathbf{u}(r_o) = \mathbf{0}. \tag{6}$$

By introducing the perturbed fields (3) and (4) in the Navier-Stokes equations (1) and neglecting nonlinear terms, we obtain the solenoidal eigenvalue problem for the (n,k) azimuthal-axial mode of the perturbation

$$\lambda u_r = {\rm D}q + \Delta_{\rm B} u_r + \left[\frac{2}{r}v_\theta^{\rm B} - \frac{2{\rm i}n}{r^2}\right] u_\theta \ , \tag{7}$$
$$\lambda u_\theta = \frac{{\rm i}n}{r}q + \Delta_{\rm B} u_\theta + \left[\frac{2{\rm i}n}{r^2} - {\rm D}_+ v_\theta^{\rm B}\right] u_r \ , \tag{8}$$
$$\lambda u_z = {\rm i}kq + \Delta_{\rm B} u_z \ , \tag{9}$$
$${\rm D}_+ u_r = -\frac{{\rm i}n}{r}u_\theta - {\rm i}k u_z \ , \tag{10}$$
$$\Delta_{\rm B} = {\rm D}_+{\rm D} - \frac{n^2+1}{r^2} - k^2 - \frac{{\rm i}n}{r}v_\theta^{\rm B}, \tag{11}$$

where ${\rm D} = \dfrac{\rm d}{{\rm d}r}$ and ${\rm D}_+ = {\rm D} + \dfrac{1}{r}$. We discretize the boundary value problem (6) - (11), is numerically discretized making use of a solenoidal Petrov-Galerkin spectral method, which has been already used in [5] for the stability analysis of spiral Couette problem. The discretization scheme finally leads to an eigenvalue problem for the amplitudes $\mathbf{a} = (a_0, \dots, a_{\rm M})^T$ of the spectral representation of the velocity field:

$$\mathbb{L}(Ri, Ro, \eta, n, k)\mathbf{a} = \lambda \mathbf{a}, \tag{12}$$

where the matrix $\mathbb{L}$ implicitly depends on the set of parameters of the boundary value problem. The linear stability problem is then reduced to the computation of the spectrum of eigenvalues of $\mathbb{L}$ for each pair of (n,k) azimuthal-axial modes. If, for a fixed set of values Ri, Ro and η, the (n,k)-spectra always lie on the left hand side of the complex plane, then the basic flow will be stable with respect to infinitesimal perturbations. On the other hand, if one of the eigenvalues of the spectrum has positive real part, then the basic Couette flow will be linearly unstable. We focus our attention in the transient evolution of perturbations in the regime of linear stability following the same methodology used in [6]. For a given (n,k) azimuthal-axial mode, consider the linear subspace ${\rm S_N}$ spanned by the eigenvectors of the N rightmost eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_{\rm N}\}$ of the spectrum of $\mathbb{L}$

$${\rm S_N} = \langle \tilde{\mathbf{q}}_1, \tilde{\mathbf{q}}_2, \dots, \tilde{\mathbf{q}}_{\rm N}\rangle. \tag{13}$$

Any perturbation $\mathbf{q}$ can be expressed as a linear combination of the eigenvectors $\tilde{\mathbf{q}}_i$

$$\mathbf{q} = \sum_{n=1}^{\rm N} \kappa_n \tilde{\mathbf{q}}_n = (\kappa_1, \kappa_2, \dots, \kappa_{\rm N})^T, \tag{14}$$

and its time evolution is dictated by the diagonal system

$$\frac{{\rm d}\kappa_i}{{\rm d}t} = \sum_{j=1}^{\rm N} \Lambda_{ij}\kappa_j, \quad (i = 1, \dots, {\rm N}), \tag{15}$$

where $\Lambda_{ij} = \lambda_i \delta_{ij}$, δ_{ij} being the Kronecker symbol. We define the *energy norm* of the perturbation $\mathbf{q}$ by means of the inner product

$$\varepsilon(\mathbf{q}) = (\mathbf{q},\mathbf{q})_{\rm E} = \frac{1}{2}\int_{r_i}^{r_o} \mathbf{q}^* \cdot \mathbf{q}\, r{\rm d}r, \tag{16}$$

where $*$ stands for the complex conjugation. For practical purposes, it is convenient to work with the standard 2-norm in the space ${\rm S_N}$

$$||\mathbf{q}||_2^2 = \sum_{j=1}^{\rm N} \kappa_j^* \kappa_j, \quad \forall \mathbf{q} \in {\rm S_N}. \tag{17}$$

We consider the matrix of inner products between the eigenvectors

$$\mathbb{M}_{ij} = (\tilde{\mathbf{q}}_i, \tilde{\mathbf{q}}_j)_{\rm E}. \tag{18}$$

This matrix is positive definite and it admits a decomposition of the form $\mathbb{M} = \mathbb{F}^\dagger \mathbb{F}$, where $\dagger$ stands for the complex conjugated transposed. The energy norm of the perturbation $\mathbf{q}$ in (16) can be expressed in the standard 2-norm in ${\rm S_N}$ by means of the components $\mathbb{F}$ and $\mathbb{F}^\dagger$:

$$\varepsilon(\mathbf{q}) = \kappa^\dagger \mathbb{M}\kappa = (\mathbb{F}\kappa, \mathbb{F}\kappa)_2 = (\kappa,\kappa)_{\rm E} = ||\kappa||_{\rm E}^2 = ||\mathbb{F}\kappa||_2^2,$$

where $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_\mathrm{N})^T$.

We are interested in the measurement of the energy growth of an initial condition κ_0 as a function of time. More specifically, we define the *energy amplification factor*, $\mathrm{g}(t)$, as the ratio between the energy norm of the perturbation at time t and its initial norm

$$\mathrm{g}(t) = \frac{||\kappa(t)||_\mathrm{E}^2}{||\kappa_0||_\mathrm{E}^2} = \frac{||\mathrm{e}^{\Lambda t}\kappa_0||_\mathrm{E}^2}{||\kappa_0||_\mathrm{E}^2}. \tag{19}$$

For a fixed time t, we want to maximize $\mathrm{g}(t)$ in (19) over the set of *all* possible initial conditions κ_0. Maximization of the ratio appearing in (19) leads to the quantity $\mathrm{G}(t)$, the *optimal energy amplification factor*

$$\mathrm{G}(t) = \max_{||\kappa_0||\neq 0} \frac{||\mathrm{e}^{\Lambda t}\kappa_0||_\mathrm{E}^2}{||\kappa_0||_\mathrm{E}^2} = ||\mathbb{F}\mathrm{e}^{\Lambda t}\mathbb{F}^{-1}||_2^2. \tag{20}$$

The quantity $||\mathbb{F}\mathrm{e}^{\Lambda t}\mathbb{F}^{-1}||_2$ is the principal singular value σ_1 of the operator $\mathbb{F}\mathrm{e}^{\Lambda t}\mathbb{F}^{-1}$ and its computation is straightforward via the SVD algorithm [7]

$$\mathrm{G}(t) = \sigma_1^2(\mathbb{F}\mathrm{e}^{\Lambda t}\mathbb{F}^{-1}). \tag{21}$$

The optimal growth $\mathrm{G}(t)$ in (21) has been obtained from the linear operator Λ associated with the (n,k) azimuthal-axial mode and for a prescribed positive time t. Therefore, for a fixed set of values Ri, Ro and η, the *maximum energy amplification factor*, G_max, is obtained by maximizing $\mathrm{G}(t)$ in (21) for all the pairs $(n,k) \in \mathbb{Z}\times\mathbb{R}$ and for $t \in \mathbb{R}^+$

$$\mathrm{G}_\mathrm{max}(Ri, Ro, \eta) = \sup_{(n,k,t)} \mathrm{G}(t). \tag{22}$$

Parametric study of G_max

In this section we describe the global features of the growth factor G_{max} defined in equation (22). The exploration has been carried out for the particular case $\eta = 0.881$ and for inner and outer Reynolds numbers in the domain $(Ri, Ro) \in [0, 900] \times [-4000, 500]$, following the specifications of the experimental study provided in [1]. Our attention is mainly focused in the counter-rotating region, where the flow exhibited subcritical transitions in the laboratory. Nevertheless, for completeness we enhanced our exploration to a small region in the co-rotating zone. In this particular exploration we have maximized the factor G in (21) for positive times, for azimuthal modes in the range $0 \le n \le 15$ and for axial wavenumbers in the range $0 \le k \le 10$. The results are summarized in figure 1. The shaded zone represents the region of the (Ro, Ri)-plane where the circular Couette is linearly unstable. Below the critical boundary prescribed by the modal analysis, we have represented the isovalues of the function $\mathrm{G}_{max}(Ro, Ri)$. Different features

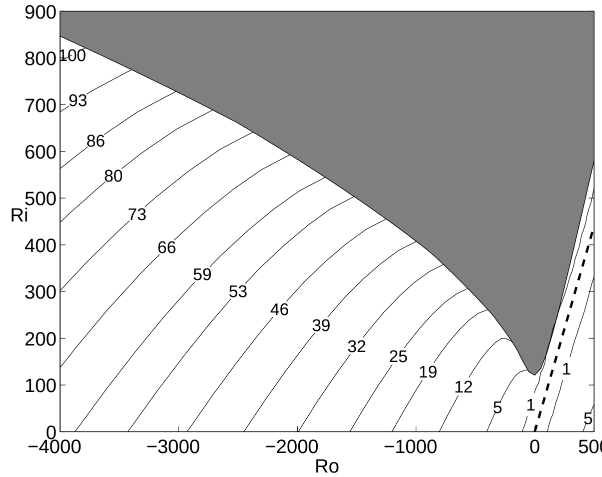

Figure 1: Maximum transient growth factor G_max in the $(Ro, Ri)-$plane. The dashed line represents the rigid body rotation curve $Ri = \eta Ro$.

can be pointed out. First, at the bottom right of figure A we have represented the usually termed *rigid rotation curve*, $Ri = \eta Ro$, by a dashed line representing the region where both cylinders rotate with the same angular speeds, $\Omega_i = \Omega_o$. We can observe that, close to that region, the Couette flow does not exhibit transient growth. This is clearly visualized in the figure by a narrow stripe containing the rigid rotation curve within which $\mathrm{G}_\mathrm{max} = 1$. This result is in agreement with previous analyses based on energy methods which concluded that near the rigid rotation region, circular Couette flow is absolutely, monotonically and globally stable, [3]. Second, in the counter-rotation region, we can observe a monotonical growth of G_{max}, which ranges between 1 and 100. When the inner cylinder is held at rest, $Ri = 0$, the factor G_max achieves a maximum value of 60 for $Ro = -4000$.

Growth mechanism and azimuthal streaks.

In this section we study how the nonnormal growth mechanism affects the basic azimuthal Couette flow. It has long been known that shear flows as plane Couette or pipe Poiseuille flow exhibit transition to secondary transient flows usually termed *streaks*. These flows are particularly easy to trigger when perturbing the basic field by means of *streamwise vortices*, i.e. vortical structures which are uniform along the direction of the basic flow. Initially, the streamwise vortices only perturb the spanwise and normal components of the flow. The *lift-up effect* is eventually responsible for the formation of the streaks by transferring the spanwise-normal contribution of the energy to the streamwise direction, [6]. Streaks are re-

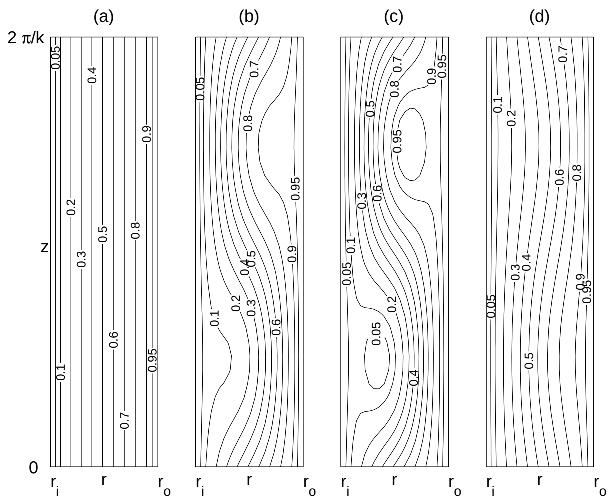

Figure 2: Isovalues of the modulated azimuthal Couette flow for $Ri = 0$, $Ro = -4000$ at different times: (a) $t = 0$, (b) $t = \tau_o/10$, (c) $t = \tau_o/5$, (d) $t = 2\tau_o$. The flow has been renormalized with respect to Ro.

gions of the fluid where the modulated flow attains high and low speeds relative to the basic flow. The modulated flow results in a profile which is, in a transient sense, linearly unstable with respect to three-dimensional perturbations. This last instability is usually termed *streak breakdown* and is one possible scenario of transition to turbulence in shear flows. In the Taylor-Couette narrow gap geometry, where the curvature is considerably reduced, the *azimuthal* coordinate plays the role of the *streamwise* direction and axisymmetric *toroidal* vector fields are suitable candidates to be streamwise vortices. Two factors are essential in order to study the time evolution of the perturbations and the modulation of the Couette flow. The first is the energy of the initial perturbation with respect to the energy of the basic flow, $\mathrm{E_B}$. This quantity is given by the expression

$$\mathrm{E_B} = \frac{A^2}{8}(r_o^4 - r_i^4) - \frac{B^2}{2}\ln\eta + \frac{AB}{2}(r_o^2 - r_i^2). \quad (23)$$

The second the time scale during which the transient streaks achieve their maximum amplitude. In our nondimensionalization, the time scale was given by the usually termed *viscous time*, $t = d^2/\nu$. We are interested in the characteristic time that a perturbation needs to reach its maximum amplitude and how this time is related to the driving dynamics of the cylinders. In counter-rotation situations, a suitable advective time scale is given by the *outer rotation period*, τ_o, which is the time that the outer cylinder needs to complete one rotation

$$\tau_o = \frac{2\pi}{Ro(\eta - 1)}, \quad Ro < 0. \quad (24)$$

In figure 2, we have plotted the isovalues of the modulated azimuthal Couette flow for $t = 0, \tau_o/10, \tau_o/5, 2\tau_o$. In this computation, the initial perturbation was an axisymmetric vortex ($n = 0$) with $k = \pi/2$, zero azimuthal component and initial energy 1.5% of $\mathrm{E_B}$. It can be clearly observed the formation of azimuthal streaks near the inner and outer cylinders.

Conclusions

A comprehensive exploration of the nonnormal transient growth in counter-rotating Taylor-Couette flow has been provided. The exploration has been done for different azimuthal and axial modes and the analysis in the radial direction presented nonmodal behaviour. Oblique modes seem to be more effective in the transient mechanism and azimuthal streaks may be observed as well although they exhibit a weaker amplification. Fully nonlinear analysis is required to understand completely the transition mechanism to turbulent regimes.

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